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Theory of Fractional Integrals and Derivatives: Application to Motion Control

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Abstract The theory of fractional derivatives and integrals (FDI's) is still in a research stage but recent progresses in the area of chaos reveal promising aspects for future developments. In the field of automatic control systems some preliminary results are restricted to the frequency domain. In this paper a novel method for the FDI approximation is presented. The proposed algorithms adopt the time domain which makes them well suited for *z*-transform analysis and digital implementation. Based on the new concepts the paper shows that classical *P*, *I* and *D* actions are special cases of a broader paradigm.

1 INTRODUCTION

The generalization of the concept of derivative of order α to non-integer values goes back to the beginning of the theory of differential calculus. In fact, Leibniz, in his correspondence with Bernoulli, L'Hôpital and Wallis (1695), had several notes about the calculation of a fractional derivative of order $\alpha = 1/2$. Nevertheless, the development of the theory of Fractional Derivatives and Integrals (FDI's) is due to Euler, Liouville. and Abel (1823). More recently, several mathematicians (Riemann (1847), Holmgren (1865), Letnikov (1868), Hadamard (1892), Weyl (1917) and Marchaud (1927)) extended the concept of FDI in several directions such as FDI's with complex values of α and fractional differential equations [1]. In the fields of physics and chemistry, FDI's are presently associated with the application of fractals in the modeling of electro-chemical reactions, irreversibility and electromagnetism [2-9]. The adoption of the theory of FDI's in control algorithms has been recently studied [10] using the Fourier transform and the frequency domain. Nevertheless, this research is still giving its first steps and further investigation is required. Moreover, the frequency-based approach has several limitations when thinking on its computer implementation.

This paper presents the fundamental aspects of the theory of FDI's and develops a novel approximation method for the direct implementation in discrete-time control algorithms. In this perspective, the paper is organized as follows. Section 2 introduces the main mathematical aspects of the theory while section 3 analyses the frequency domain approximation to FDI's. Section 4 develops a new procedure for the implementation of FDI's in control system design. The new method consists on a discrete-time approximation that leads, directly, to *z*-domain formulae well suited for digital algorithms. Based on this method, section 5 studies the application of FDI's in motion control and compares the results with classical *P*, *I* and *D* actions. Finally, in section 6, the main conclusions are drawn.

2 MATHEMATICAL ASPECTS OF THE THEORY OF FRACTIONAL DERIVATIVES AND INTEGRALS

The mathematical definition of a derivative or integral of fractional order has been the subject of several different approaches. For example, a "direct" definition based on the concept of fractional differential $\Delta_h^\alpha x(t)$ of order α, is due to Letnikov and leads to the expression for $D^\alpha x(t)$, the fractional derivative of order α:

$$D^\alpha x(t) = \lim_{h \to 0} \frac{\Delta_h^\alpha x(t)}{h^\alpha} = \lim_{h \to 0} \left[\frac{1}{h^\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} x(t - kh) \right] \quad (1a)$$

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} \quad (1b)$$

where Γ is the gamma function and h is the time increment. Alternatively, adopting the Laplace operator L, the fractional derivative $D^\alpha x(t)$ and the fractional integral $I^\alpha x(t)$ of order $\alpha \in C$ obey the definitions:

$$D^\alpha x(t) = L^{-1}\{s^\alpha X(s)\} \quad (2a)$$

$$I^\alpha x(t) = L^{-1}\{s^{-\alpha} X(s)\} \quad (2b)$$

where $X(s) = L\{x(t)\}$. Based on these definitions it is possible to find the general expressions for the FDI's of several standard functions, such as those depicted on Table 1.

3 FREQUENCY-BASED APPROXIMATION TO FDI'S

The application of FDI's is presently an area of leading research [9] due to the development of the chaos theory. In what concerns automatic control systems, we must mention the pioneer work of Oustaloup [8,10] that studied the application of FDI's from the point of view of the frequency response.

Table 1 Integrals and Derivatives of Fractional Order

$\varphi(x), x \in \Re$	$\left(I_{+}^{\alpha}\varphi\right)(x), x \in \Re, \alpha \in C$
$(x-a)^{\beta-1}$	$\frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}, \mathrm{Re}(\beta)>0$
$e^{\lambda x}$	$\lambda^{-\alpha}e^{\lambda x}, \mathrm{Re}(\lambda)>0$
$\begin{cases}\sin(\lambda x)\\ \cos(\lambda x)\end{cases}$	$\lambda^{-\alpha}\begin{cases}\sin(\lambda x-\alpha\pi/2)\\ \cos(\lambda x-\alpha\pi/2)\end{cases}, \lambda>0, \mathrm{Re}(\alpha)>1$
$e^{\lambda x}\begin{cases}\sin(\gamma x)\\ \cos(\gamma x)\end{cases}$	$\frac{e^{\lambda x}}{\left(\lambda^2+\gamma^2\right)^{\alpha/2}}\begin{cases}\sin(\gamma x-\alpha\phi)\\ \cos(\gamma x-\alpha\phi)\end{cases}, \begin{matrix}\phi=\arctan(\gamma/\lambda)\\ \gamma>0, \mathrm{Re}(\lambda)>1\end{matrix}$

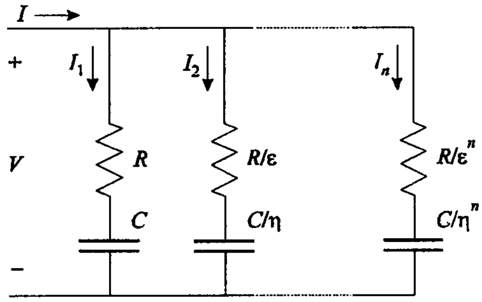

Figure 1: Electrical circuit with recursive association of resistance and capacitance elements.

In order to analyze the frequency-based approach to the FDI theory, let us consider the recursive circuit represented on Figure 1 such that:

$$I=\sum_{i=1}^{n} I_i\ ,\ R_{i+1}=\frac{R_i}{\varepsilon}\ ,\ C_{i+1}=\frac{C_i}{\eta} \tag{3}$$

where η and ε are scale factors, I is the current due to an applied voltage V and R_i and C_i are the resistance and capacitance elements of the ith branch of the circuit. Therefore, the frequency-dependent admittance $Y(j\omega)$ is given by:

$$I(j\omega)=V(j\omega)\cdot Y(j\omega) \tag{4a}$$

$$Y(j\omega)=\sum_{i=0}^{n}\frac{j\omega C\varepsilon^{i}}{j\omega CR+(\eta\varepsilon)^{i}} \tag{4b}$$

Figure 2 shows the asymptotic Bode diagrams of amplitude and phase of $Y(j\omega)$. The pole and zero frequencies (ω_i and ω_i') obey the recursive relationships:

$$\frac{\omega_{i+1}'}{\omega_i'}=\frac{\omega_{i+1}}{\omega_i}=\varepsilon\eta\ ,\ \frac{\omega_i}{\omega_i'}=\varepsilon\ ,\ \frac{\omega_{i+1}'}{\omega_i}=\eta \tag{5}$$

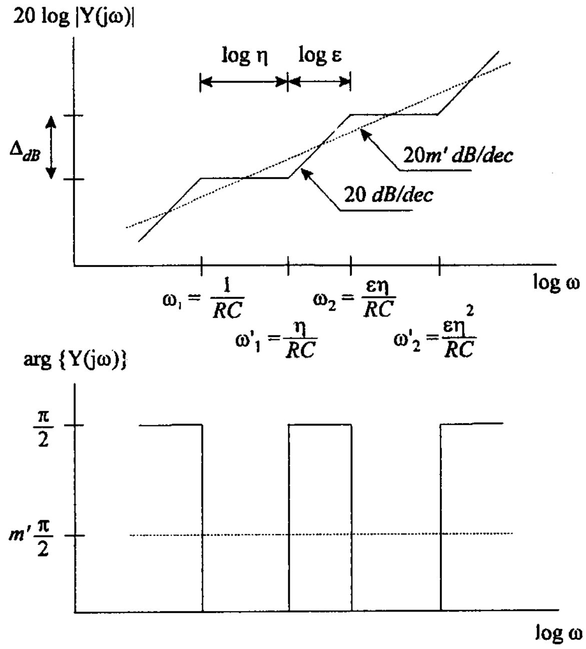

Figure 2: Asymptotic Bode diagrams of amplitude and phase of $Y(j\omega)$.

For the Bode diagram of amplitude, the average slope m' can be calculated according with the expressions:

$$\left.\begin{aligned}20m'\ dB/dec&=\frac{\Delta_{dB}}{\log\varepsilon+\log\eta}\\ 20\ dB/dec&=\frac{\Delta_{dB}}{\log\varepsilon}\end{aligned}\right\}\Rightarrow m'=\frac{\log\varepsilon}{\log\varepsilon+\log\eta} \tag{6}$$

Alternatively, for the Bode diagram of phase, tacking the average leads to a similar result:

$$m'\frac{\pi}{2}(\log\varepsilon+\log\eta)=\frac{\pi}{2}\log\varepsilon \tag{7}$$

The fractional order of the frequency response is due to the recursive nature of the circuit. In fact, the admittance $Y(j\omega)$ follows the recursive formula:

$$Y\left(\frac{\omega}{\eta\varepsilon}\right)=\frac{1}{\varepsilon}Y(\omega) \tag{8}$$

with solution (K is a scale factor) in accordance with (6):

$$Y(\omega)=K(j\omega)^{-m'}\ ,\ m'=\frac{\log\varepsilon}{\log\varepsilon+\log\eta} \tag{9}$$

Another important aspect of the FDI can be illustrated through the elemental control system represented in Figure 3, where $1<\alpha<2$. The system open-loop frequency response and the root locus are depicted in Figure 4.

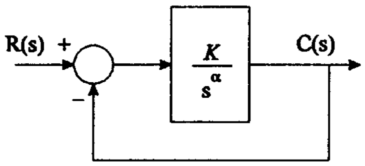

Figure 3: Block diagram for an elemental feedback control system of fractional order α.

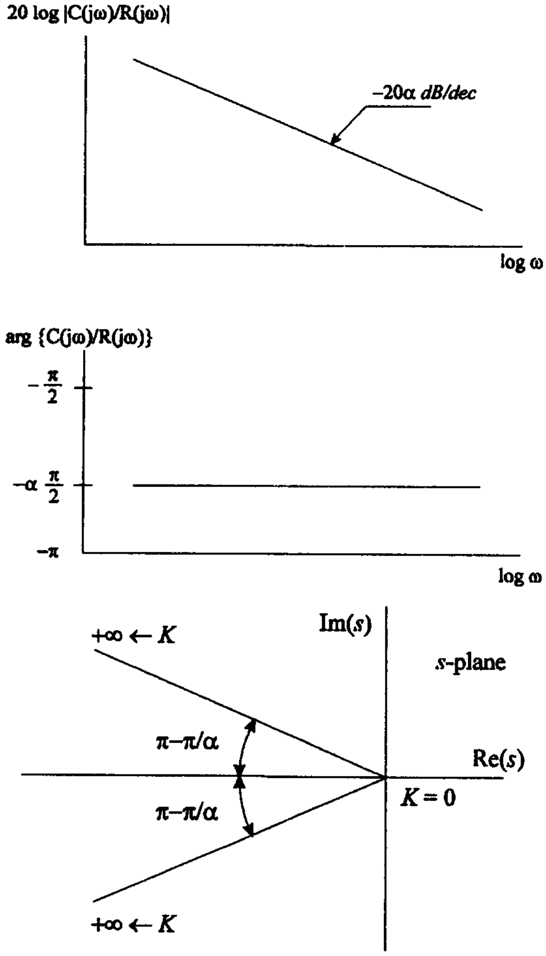

Figure 4: Bode diagram and root locus stability analysis for a feedback control system of fractional order $1 < \alpha < 2$.

Both representations reveal that the fractional order control system is robust against gain variations, maintaining a constant stability margin, in contrast with classical integer order control systems. These conclusions highlight the importance of FDI's in control system theory. Nevertheless, the frequency-based approach limits the areas of application and requires additional calculation procedures for the implementation of discrete-time algorithms.

4. Discrete-Time Approximation to FDI's

The frequency-domain approach for the FDI implementation leads to a finite number of poles and zeros, established through recursive formulae such as equations (5). Nevertheless, this method has several drawbacks:

- The bandwidth of the FDI approximation is restricted to a limited range
- The finite number of poles and zeros yields a "ripple" in the frequency response
- The conversion $s \to z$ (i.e. the transformation to the discrete-time domain) requires further calculations and, possibly, additional approximations.

To overcome these problems this section presents a new method for the development of FDI-based discrete-time control systems [11]. In order to implement, directly, FDI's in the z domain we adopt three types of approximation:

- Truncation and discrete-time evaluation of the series resulting from the FDI definition
- First order (i.e. linear) function interpolation
- Second order (i.e. quadratic) function interpolation

In the following sub-sections these approximations are studied and the results compared.

4.1 Truncation of the FDI definition series

As referred previously, a FDI can be obtained through the series defined in (1). Therefore, for a discrete-time control algorithm with sampling period T, this formula can be approximated through a n-th order truncated series, resulting the following equations in the time and z domains:

$$D^{\alpha}x(t) \approx \frac{1}{T^{\alpha}} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} x(t-kT) \tag{10a}$$

$$Z\left\{D^{\alpha}x(t)\right\} \approx \left\{ \frac{1}{T^{\alpha}} \sum_{k=0}^{n} (-1)^k \frac{\Gamma(\alpha+1) z^{-k}}{k!\,\Gamma(\alpha-k+1)} \right\} X(z) \tag{10b}$$

Clearly, in order to have good approximations, we must have a large number of terms and a small sampling period.

4.2 First order function interpolation

A FDI can be approximated through a linear interpolation based on the two last sampled values. Therefore, interpolating $x(k-1)$ and $x(k)$ in $0 \le t \le T$ results:

$$x(t) = \left[x(k) - x(k-1)\right]\frac{t}{T} + x(k-1) \tag{11}$$

The integral I^{α} of fractional order α, is given by:

$$I^{\alpha}x(t) = \frac{x(k) - x(k-1)}{\Gamma(2+\alpha)} \cdot \frac{t^{\alpha+1}}{T} + \frac{x(k-1)}{\Gamma(1+\alpha)} t^{\alpha} \tag{12}$$

Consequently, for $t = T$ the time and z-domain formulae are:

$$I^{\alpha}x(t)\Big|_{t=T} = \frac{T^{\alpha}}{\Gamma(2+\alpha)}\left[x(k) + \alpha \cdot x(k-1)\right] \tag{13a}$$

$$Z\{I^{\alpha}x(t)\} = \frac{T^{\alpha}}{\Gamma(2+\alpha)}\left(1+\alpha\cdot z^{-1}\right)\cdot X(z) \quad (13b)$$

For $\alpha = -1, 0, 1$ these expressions correspond to the differential (*D*), proportional (*P*) and integral (*I*) actions, respectively, given by the well known equations:

$$H_D(z) = \frac{1}{T}\left(1-z^{-1}\right) \quad (14a)$$

$$H_P(z) = 1 \quad (14b)$$

$$H_I(z) = \frac{T}{2}\left(1+z^{-1}\right) \quad (14c)$$

In this perspective, for a first-order interpolation, I^{α} can be interpreted as the sum of *PD* or *PI* actions with gains K_P, K_D or K_P, K_I, respectively. For the two cases we get:

$$\frac{T^{\alpha}}{\Gamma(2+\alpha)}\left(1+\alpha\cdot z^{-1}\right) = K_P\cdot H_P(z) + K_D\cdot H_D(z) \quad (15a)$$

$$\frac{T^{\alpha}}{\Gamma(2+\alpha)}\left(1+\alpha\cdot z^{-1}\right) = K_P\cdot H_P(z) + K_I\cdot H_I(z) \quad (15b)$$

$$K_P = \frac{(1+\alpha)T^{\alpha}}{\Gamma(2+\alpha)},\ K_D = -\frac{\alpha T^{\alpha+1}}{\Gamma(2+\alpha)} \quad (16a)$$

$$K_P = \frac{(1-\alpha)T^{\alpha}}{\Gamma(2+\alpha)},\ K_I = \frac{2\alpha T^{\alpha-1}}{\Gamma(2+\alpha)} \quad (16b)$$

These equations reveal that the *D* and *I* actions have opposite effects, resulting:

- $\{K_D > 0, K_I < 0\}$ for $-1 < \alpha < 0$
- $\{K_D < 0, K_I > 0\}$ for $0 < \alpha < 1$.

4.3 Second order function interpolation

Another FDI approximation consists on a quadratic interpolation. Therefore, interpolating $x(k-2)$, $x(k-1)$ and $x(k)$ in the interval $0 \le t \le 2T$, results:

$$x(t) = \frac{x(k)-2x(k-1)+x(k-2)}{2}\left(\frac{t}{T}\right)^2 - \\ -\left[x(k)-4x(k-1)+3x(k-2)\right]\frac{t}{T} + x(k-2) \quad (17)$$

In the line of thought established in the previous sub-section, we can get I^{α} as:

$$I^{\alpha}x(t)\Big|_{t=2T} = \frac{2^{\alpha}T^{\alpha}}{\Gamma(3+\alpha)}\Big[(2-\alpha)\cdot x(k) + \\ +4\alpha\cdot x(k-1) + \alpha^2\cdot x(k-2)\Big] \quad (18a)$$

$$Z\{I^{\alpha}x(t)\} = \frac{2^{\alpha}T^{\alpha}}{\Gamma(3+\alpha)}\left[(2-\alpha)+4\alpha z^{-1}+\alpha^2 z^{-2}\right]X(z) \quad (18b)$$

For $\alpha = -1, 0, 1$ equations (18) yield the *D*, *P* and *I* actions:

$$H_D(z) = \frac{1}{T}\left(\frac{3}{2} - 2z^{-1} + \frac{1}{2}z^{-2}\right) \quad (19a)$$

$$H_P(z) = 1 \quad (19b)$$

$$H_I(z) = \frac{T}{3}\left(1+4z^{-1}+z^{-2}\right) \quad (19c)$$

Consequently, for a second-order interpolation, I^{α} can be interpreted as the sum of *PID* actions with gains K_P, K_I and K_D, that is:

$$\frac{2^{\alpha}T^{\alpha}}{\Gamma(3+\alpha)}\left[(2-\alpha)+4\alpha\cdot z^{-1}+\alpha^2\cdot z^{-2}\right] = \\ = K_P\cdot H_P(z) + K_I\cdot H_I(z) + K_D\cdot H_D(z) \quad (20)$$

$$K_P = \frac{\left(1-\alpha^2\right)2^{\alpha+1}T^{\alpha}}{\Gamma(3+\alpha)} \quad (21a)$$

$$K_I = \frac{3\alpha(1+\alpha)2^{\alpha-1}T^{\alpha-1}}{\Gamma(3+\alpha)} \quad (21b)$$

$$K_D = \frac{\alpha(\alpha-1)2^{\alpha}T^{\alpha+1}}{\Gamma(3+\alpha)} \quad (21c)$$

The *D* and *I* actions have opposite effects, but similar "magnitudes", for symmetric values of α resulting:

- $\{K_D > 0, K_I < 0\}$ for $-1 < \alpha < 0$
- $\{K_D < 0, K_I > 0\}$ for $0 < \alpha < 1$.

5 APPLICATION OF THE THEORY OF FDI'S TO MOTION CONTROL SYSTEMS

A simple mass may be considered as a prototype mechanical system. Therefore, in order to study the performances of an elemental control system, for fractional values of α, we consider a mass *M* with model:

$$Z\left\{\frac{1}{Ms^2}\right\} = \frac{T^2}{2M}\frac{z+1}{(z-1)^2} \quad (22)$$

The elemental control algorithm consists on an action of fractional order that results for $\alpha = 1/2$ and $\alpha = -1/2$. Figures 5, 6 and 7 show the root locus, in the *z*-domain, for $D^{1/2}$ and $I^{1/2}$ and the three types of approximations. The results show that, with the adoption of FDI's, we have a continuous variation of the control action in contrast with the "discrete steps" available in the classical *D*, *P* or *I* algorithms.

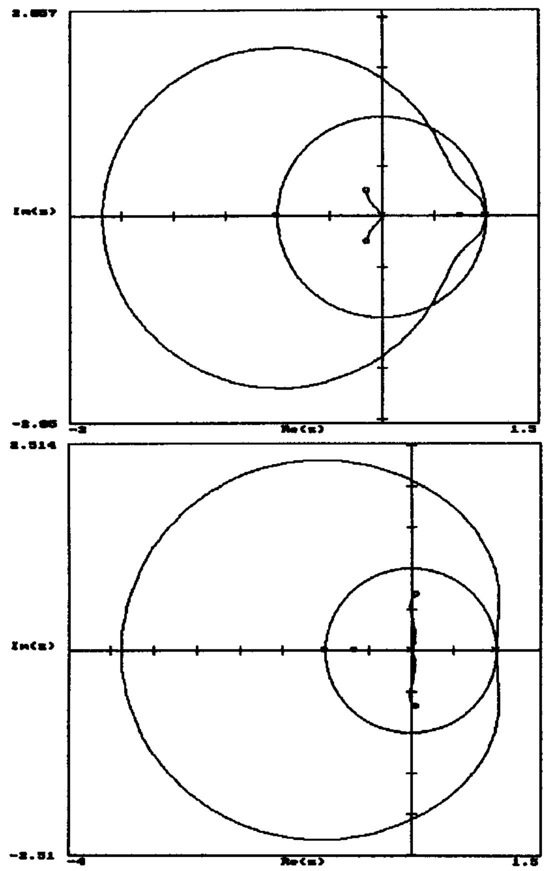

Figure 5: *Z*-domain root locus for a mass and control actions $D^{1/2}$ and $I^{1/2}$ based on a 3-th order series approximation.

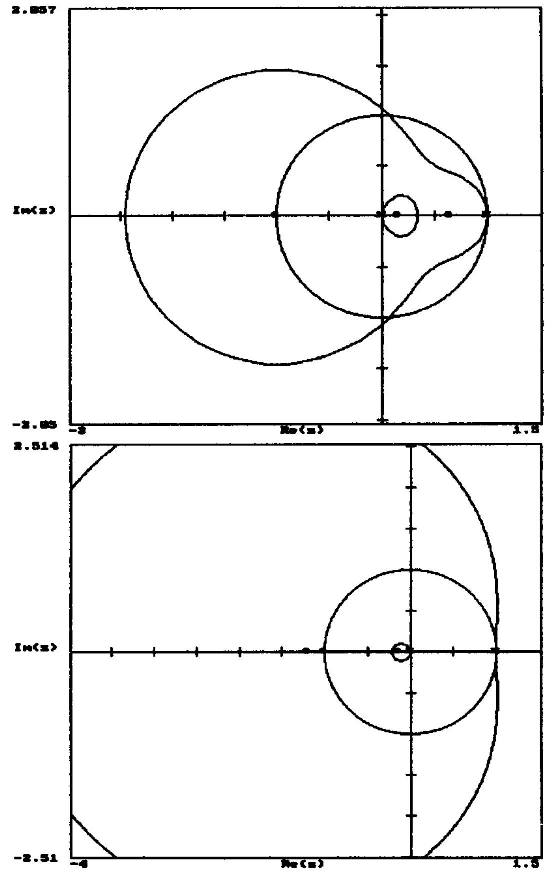

Figure 7: *Z*-domain root locus for a mass and control actions $D^{1/2}$ and $I^{1/2}$ based on a quadratic interpolation.

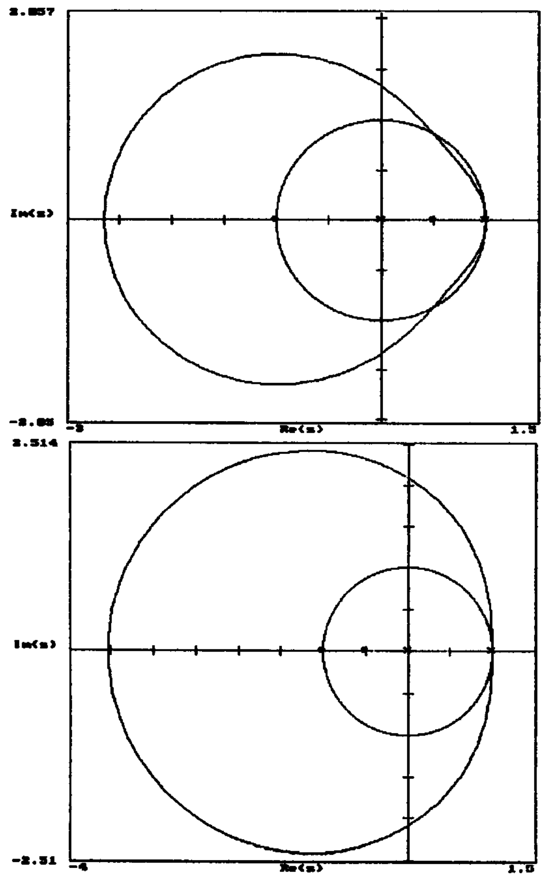

Figure 6: *Z*-domain root locus for a mass and control actions $D^{1/2}$ and $I^{1/2}$ based on a linear interpolation.

In order to investigate the robustness of the FDI-based control algorithms we introduced a nonlinear block in the forward path (Figure 8), corresponding to a saturation phenomena in the actuator.

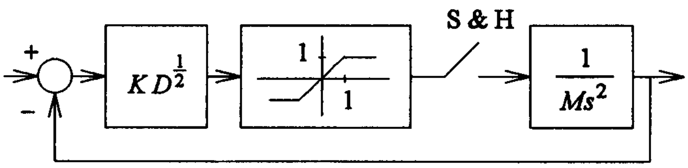

Figure 8: FDI control system a of a mass system with actuator saturation.

Figure 9 shows the system response, for a unity step input, without actuator saturation, for $K = 10$, $M = 1$, a sample and hold time $T = 0.1$ and a n-th order approximation to $D^{1/2}$ according with (10). It is clear that the higher the order of the approximation the better the response. Therefore, the *PD* like scheme, that is equivalent to the approximation of order $n = 1$, is inferior to the fractional order controller. Furthermore, the robustness of the fractional order algorithm over classical control actions is highlighted in the presence of a saturation phenomena. Figure 10 shows the system response in this case for a n-th order approximation. The *PD*-like (i.e. for $n = 1$) response is very sensitive to the saturation effect while the $D^{1/2}$ controller remains highly stable.

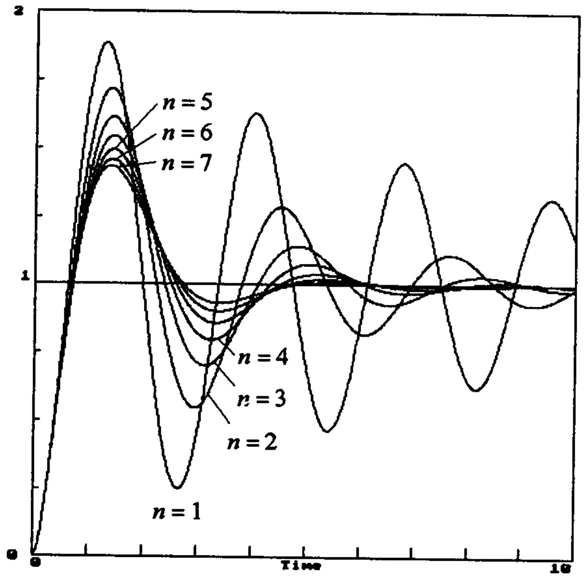

Figure 9: System response, without saturation, for $K = 10\,M$ and a n-th order series approximation to $D^{1/2}$.

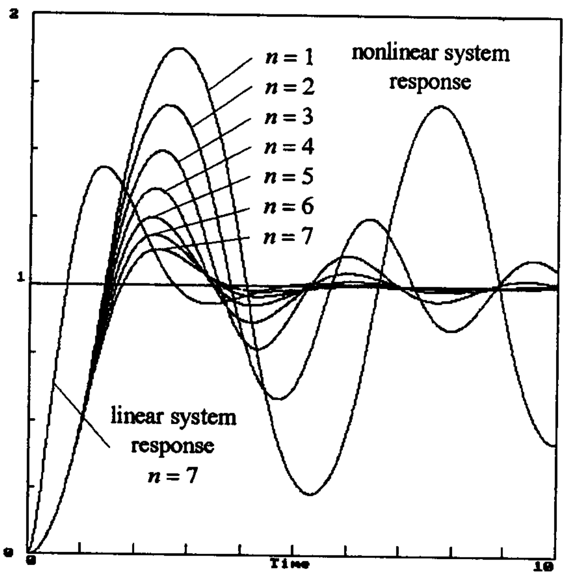

Figure 10: System response, with saturation, for $K = 10\,M$ and a n-th order series approximation to $D^{1/2}$.

6 CONCLUSIONS

The theory of FDI's is still in a research stage but the recent progress in the areas of chaos and fractal reveals promising aspects for future developments. In the field of automatic control systems some preliminary work has been carried out but the results are restricted to the frequency domain.

In this paper a novel method for the FDI approximation was presented. The proposed algorithms adopt the time domain which makes them well suited for z-transform analysis and microcomputer implementation. For a prototype mechanical system the control algorithm based on the new concepts reveals that classical P, I and D actions are, in fact, special cases of a more broad paradigm. Moreover, the fractional controllers are very robust to gain variations that arise in non-linear systems. This behavior makes FDI-based algorithms that are well suited for application in mechatronic where non-linear dynamic phenomena are usually present. In this line of thought, this study represents a first stage towards the development of motion control systems based on the theory of FDI's.

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