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ANALYSIS AND DESIGN OF FRACTIONAL-ORDER DIGITAL CONTROL SYSTEMS

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The paper presents a novel method for the design of fractional-order digital controllers. The theory of fractional derivatives and integrals (FDI's) is still in a research stage but the recent progress in the areas of chaos and fractal reveals promising aspects for future developments. In the field of automatic control systems some preliminary work has been carried out but the results are restricted to the frequency domain. The algorithms proposed in the paper adopt the time domain, which makes them well suited for z-transform analysis and digital implementation. For a prototype mechanical system the control algorithm based on the new concepts reveals that classical P, I and D actions are, in fact, special cases of a more broad paradigm. In this line of thought, this study represents a first stage towards the development of motion control systems based on the theory of FDI's.

Keywords: Controller design; fractional derivatives; digital control; fractals; chaos

1. INTRODUCTION

The generalization of the concept of derivative $D^{\alpha}f(x)$ to non-integer values of α goes back to the beginning of the theory of differential calculus. In fact, Leibniz, in his correspondence with Bernoulli, L'Hôpital and Wallis (1695), had several notes about the calculation of $D^{1/2}f(x)$. Nevertheless, the development of the theory of Fractional Derivatives and Integrals (FDI's) is due to Euler, Liouville. and Abel (1823). More recently, several mathematicians (Riemann (1847), Holmgren (1865), Letnikov (1868), Hadamard (1892), Weyl (1917) and

Marchaud (1927)) extended the concept of FDI in several directions such as FDI's with complex values of α and fractional differential equations [1]. In the fields of physics and chemistry, FDI's are presently associated with the application of fractals in the modeling of electro-chemical reactions, irreversibility and electromagnetism [2–8]. The adoption of the theory of FDI's in control algorithms has been recently studied [10] using the frequency domain. Nevertheless, this research is still giving its first steps and further investigation is required. Moreover, the frequency-based approach has several limitations when thinking on a microcomputer implementation. This paper presents the fundamental aspects of the theory of FDI's and develops a novel approximation method for the direct implementation in discrete-time control algorithms [11, 12]. In this perspective, the paper is organized as follows. Section 2 introduces the main mathematical aspects of the theory while section 3 analyses the frequency domain approximation to FDI's. Section 4 develops a new procedure for the implementation of FDI's in control system design. The new method consists on a discrete-time approximation that leads, directly, to z-domain formulae well suited for digital algorithms. Based on this method, section 5 studies the application of FDI's in motion control and compares the results with classical PID actions. Finally, in section 6, conclusions are drawn.

2. MAIN MATHEMATICAL ASPECTS OF THE THEORY OF FDI'S

The mathematical definition of a derivative or integral of fractional order has been the subject of several different approaches. For example, a definition based on the concept of fractional differential of order α leads to $D^{\alpha}x(t)$, the fractional derivative of order α:

$$D^{\alpha}x(t) = \lim_{h \to 0}\left[\frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}(-1)^{k}\binom{\alpha}{k}x(t-kh)\right] \tag{1a}$$

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} \tag{1b}$$

where Γ is the gamma function and h is the time increment. Nevertheless, $D^{\alpha}x(t)$ can be obtained through other methods, namely using the Laplace or the Fourier transforms. In fact, adopting the Laplace operator L, the fractional derivative $D^{\alpha}x(t)$ and the fractional integral $I^{\alpha}x(t)$ of order $\alpha \in C$ obey the alternative definitions:

$$D^{\alpha}x(t) = L^{-1}\{s^{\alpha}X(s)\}, \quad I^{\alpha}x(t) = L^{-1}\{s^{-\alpha}X(s)\} \tag{2}$$

where $X(s) = L\{x(t)\}$. Based on these definitions it is possible to calculate the FDI's of several standard functions, such as those depicted on Table I.

3. FREQUENCY-DOMAIN APPROXIMATION TO FDI'S

The application of FDI's is presently a leading area of research [9] due to the development of the chaos theory. In what concerns automatic control, we must mention the pioneer work of Oustaloup [8, 10] that studied the application of FDI's from the point of view of the frequency response.

In order to analyze the frequency-based approach to the FDI theory, let us consider the recursive circuit represented on Figure 1 such that:

TABLE I Formulae of several FDI's

$\varphi(x), x \in \Re$	$(I_{+}^{\alpha}\varphi)(x), x \in \Re, \alpha \in C$
$(x-a)^{\beta-1}$	$\dfrac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}, \mathrm{Re}(\beta) > 0$
$e^{\lambda x}$	$\lambda^{-\alpha}e^{\lambda x}, \mathrm{Re}(\lambda) > 0$
$\begin{cases}\sin(\lambda x)\\ \cos(\lambda x)\end{cases}$	$\lambda^{-\alpha}\begin{cases}\sin(\lambda x - \alpha\pi/2)\\ \cos(\lambda x - \alpha\pi/2)\end{cases}, \lambda > 0, \mathrm{Re}(\alpha) > 1$
$e^{\lambda x}\begin{cases}\sin(\gamma x)\\ \cos(\gamma x)\end{cases}$	$\dfrac{e^{\lambda x}}{(\lambda^2+\gamma^2)^{\alpha/2}}\begin{cases}\sin(\gamma x - \alpha\phi)\\ \cos(\gamma x - \alpha\phi)\end{cases}, \begin{matrix}\phi = \arctan(\gamma/\lambda)\\ \gamma > 0, \mathrm{Re}(\lambda) > 1\end{matrix}$

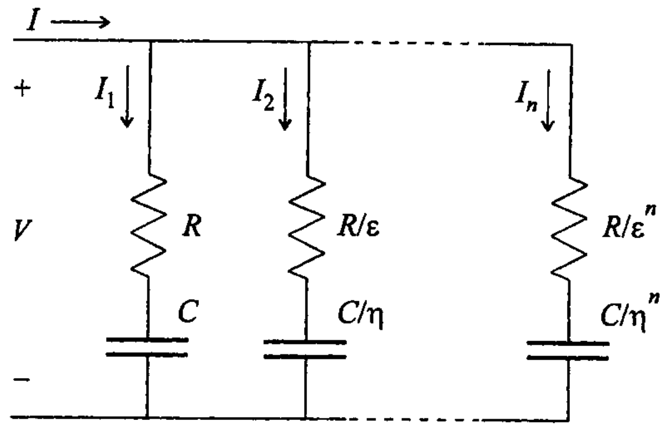


FIGURE 1 Electrical circuit with recursive association of lumped resistance and capacitance elements.

$$I=\sum_{i=1}^{n} I_i, \quad R_{i+1}=\frac{R_i}{\varepsilon}, \quad C_{i+1}=\frac{C_i}{\eta} \tag{3}$$

where η and ε are scale factors, I is the current due to an applied voltage V and R_i and C_i are the resistance and capacitance elements of the ith branch of the circuit.

The admittance $Y(j\omega)$ is given by:

$$I(j\omega)=V(j\omega)\cdot Y(j\omega), \quad Y(j\omega)=\sum_{i=0}^{n}\frac{j\omega C\varepsilon^i}{j\omega CR+(\eta\varepsilon)^i} \tag{4}$$

Figure 2 shows the asymptotic Bode diagrams of amplitude and phase of $Y(j\omega)$. The pole and zero frequencies (ω_i and ω_i') obey the recursive relationships:

$$\frac{\omega_{i+1}'}{\omega_i'}=\frac{\omega_{i+1}}{\omega_i}=\varepsilon\eta, \quad \frac{\omega_i}{\omega_i'}=\varepsilon, \quad \frac{\omega_{i+1}'}{\omega_i'}=\eta \tag{5}$$

From the Bode diagrams (of amplitude or of phase), the average slope m' can be calculated as:

$$m'=\frac{\log\varepsilon}{\log\varepsilon+\log\eta} \tag{6}$$

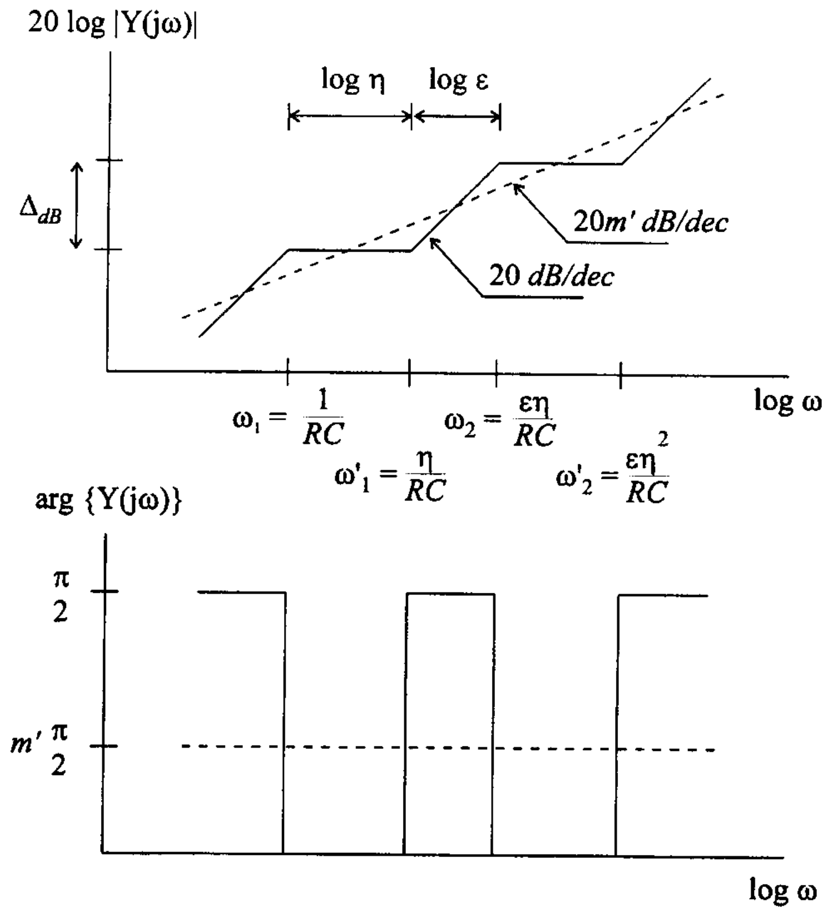


FIGURE 2 Bode diagrams of amplitude and phase of $Y(j\omega)$.

The fractional order of the frequency response is due to the recursive nature of the circuit. In fact, the admittance $Y(j\omega)$ follows the recursive formula (K is a scale factor):

$$Y\left(\frac{\omega}{\eta\varepsilon}\right) = \frac{1}{\varepsilon} Y(\omega) \tag{7}$$

with solution in accordance with (6):

$$Y(\omega) = K(j\omega)^{-m'}, \quad m' = \frac{\log \varepsilon}{\log \varepsilon + \log \eta} \tag{8}$$

Another important aspect of the FDI application can be illustrated through the elemental system represented in Figure 3, where $1 < \alpha < 2$. The system open-loop frequency response and the root locus are depicted in Figure 4.

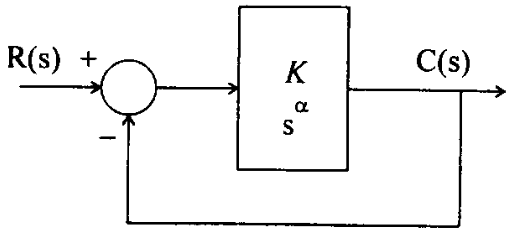


FIGURE 3 Block diagram for an elemental feedback control system of fractional order α.

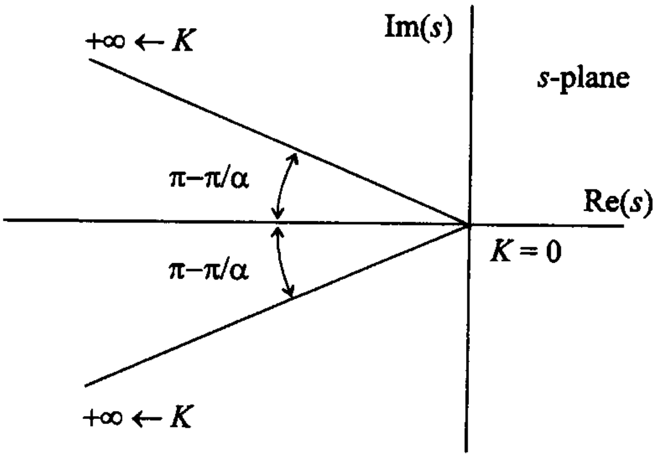


FIGURE 4 Root locus for a feedback control system of fractional order $1 < \alpha < 2$.

Both representations reveal that the fractional order control system is robust against gain variations, maintaining a constant stability margin, in contrast with classical integer order control systems.

These conclusions highlight the importance of FDI's in control system theory. Nevertheless, the frequency approach limits the areas of application and requires additional calculation procedures for the implementation of discrete-time algorithms.

4. DISCRETE-TIME APPROXIMATION TO FDI'S

The frequency-domain approach for the FDI implementation leads to a finite number of poles and zeros, established through recursive formulae such as equations (5). Nevertheless, this method has several drawbacks:

- The bandwidth of the FDI approximation is restricted to a limited range

- The finite number of poles and zeros yields a "ripple" in the frequency response
- The conversion $s \to z$ (i.e. the transformation to the discrete-time domain) requires further calculations and, possibly, additional approximations.

To overcome these problems this section presents a new method for the development of FDI-based discrete-time control systems. In order to implement, directly, FDI's in the z domain we adopt three types of approximation:

- Truncation and discrete-time evaluation of the series resulting from the FDI definition
- First order (i.e. linear) function interpolation
- Second order (i.e. quadratic) function interpolation

These approximations are studied in the next sub-sections.

4.1. Truncation of the FDI Definition Series

As referred previously, a FDI can be obtained through the series defined in (1). Therefore, for a discrete-time control algorithm with sampling period T, this formula can be approximated through a n-term truncated series, resulting the following equations in the time and z domains:

$$D^{\alpha} x(t) \approx \frac{1}{T^{\alpha}} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} x(t - kT) \tag{9a}$$

$$Z\{D^{\alpha} x(t)\} \approx \left\{ \frac{1}{T^{\alpha}} \sum_{k=0}^{n} \frac{(-1)^k \Gamma(\alpha+1)}{k!\,\Gamma(\alpha-k+1)} z^{-k} \right\} X(z) \tag{9b}$$

Clearly, in order to have good approximations, we must have a large number of terms and a small sampling period.

4.2. First Order Function Interpolation

A FDI can be approximated through a linear interpolation based on the last sampled values. Therefore, interpolating $x(k-1)$ and $x(k)$ in the interval $0 \leqslant t \leqslant T$ results:

$$x(t) = [x(k) - x(k-1)]\frac{t}{T} + x(k-1) \tag{10}$$

The integral I^α of fractional order α, is given by:

$$I^\alpha x(t) = \frac{x(k) - x(k-1)}{\Gamma(2+\alpha)} \cdot \frac{t^{\alpha+1}}{T} + \frac{x(k-1)}{\Gamma(1+\alpha)} t^\alpha \tag{11}$$

For $t = T$ the time and z-domain formulae are:

$$I^\alpha x = \frac{T^\alpha}{\Gamma(2+\alpha)}[x(k) + \alpha \cdot x(k-1)] \tag{12a}$$

$$Z\{I^\alpha x\} = \frac{T^\alpha}{\Gamma(2+\alpha)}(1 + \alpha \cdot z^{-1}) \cdot X(z) \tag{12b}$$

For $\alpha = -1, 0, 1$ these expressions correspond to the differential (D), proportional (P) and integral (I) actions, respectively, given by the well known equations:

$$H_D(z) = \frac{1}{T}(1 - z^{-1}), \quad H_p(z) = 1, \quad H_1(z) = \frac{T}{2}(1 + z^{-1}) \tag{13}$$

In this perspective, I^α given by (12) can be interpreted as the sum of PD or PI actions with gains K_P, K_D or K_P, K_I, respectively. For these two cases we get:

$$Z\{I^\alpha x\} = K_P \cdot H_P(z) + K_D \cdot H_D(z) \tag{14a}$$

$$Z\{I^\alpha x\} = K_P \cdot H_P(z) + K_I \cdot H_I(z) \tag{14b}$$

which lead to:

$$K_P = \frac{(1+\alpha)T^\alpha}{\Gamma(2+\alpha)}, \quad K_D = -\frac{\alpha T^{\alpha+1}}{\Gamma(2+\alpha)} \tag{15a}$$

$$K_P = \frac{(1-\alpha)\, T^{\alpha}}{\Gamma(2+\alpha)}, \quad K_I = \frac{2\alpha T^{\alpha-1}}{\Gamma(2+\alpha)} \tag{15b}$$

As expected, these equations reveal that the D and I actions have opposite effects, resulting $\{K_D > 0, K_I < 0\}$ for $-1 < \alpha < 0$ and $\{K_D < 0, K_I > 0\}$ for $0 < \alpha < 1$.

4.3. Second Order Function Interpolation

Another FDI approximation consists on a quadratic interpolation. Therefore, interpolating $x(k-2), x(k-1)$ and $x(k)$ in the interval $0 \leqslant t \leqslant 2T$, results:

$$x(t) = \frac{x(k) - 2x(k-1) + x(k-2)}{2}\left(\frac{t}{T}\right)^2 - [x(k) - 4x(k-1) + 3x(k-2)]\frac{t}{T} + x(k-2) \tag{16}$$

As established in the previous sub-section, we can get I^{α} for $t = 2T$ as:

$$I^{\alpha} x = \frac{2^{\alpha}\, T^{\alpha}}{\Gamma(3+\alpha)}[(2-\alpha)\, x(k) + 4\alpha x(k-1) + \alpha^2 x(k-2)] \tag{17a}$$

$$Z\{I^{\alpha} x\} = \frac{2^{\alpha}\, T^{\alpha}}{\Gamma(3+\alpha)}[(2-\alpha) + 4\alpha z^{-1} + \alpha^2 z^{-2}]X(z) \tag{17b}$$

If $\alpha = -1, 0, 1$ expressions (17) give the D, P and I actions:

$$H_D = \frac{1}{T}\left(\frac{3}{2} - 2z^{-1} + \frac{z^{-2}}{2}\right), \quad H_P = 1, \quad H_1 = \frac{T}{3}(1 + 4z^{-1} + z^{-2}) \tag{18}$$

Consequently, equation (17b) can be interpreted as the sum of *PID* actions with gains K_P, K_I and K_D, that is

$$Z\{I^{\alpha} x\} = K_P H_P + K_I H_I + K_D H_D \tag{19}$$

$$K_P = \frac{(1-\alpha^2)\, 2^{\alpha+1}\, T^{\alpha}}{\Gamma(3+\alpha)}, \quad K_I = -\frac{3\alpha(1+\alpha)\, 2^{\alpha-1}\, T^{\alpha-1}}{\Gamma(3+\alpha)},$$

$$K_D = \frac{\alpha(\alpha - 1) 2^{\alpha} T^{\alpha + 1}}{\Gamma(3 + \alpha)} \tag{20}$$

Again we conclude that the D and I actions have opposite effects, resulting $\{K_D > 0, K_I < 0\}$ for $-1 < \alpha < 0$ and $\{K_D < 0, K_I > 0\}$ for $0 < \alpha < 1$. Furthermore, the D and I actions reveal similar "weights" for opposite values of α.

5. APPLICATION OF FDI'S TO MOTION CONTROL SYSTEMS

A simple mass may be considered as a prototype mechanical system. Therefore, in order to study the performances of an elemental control system, for fractional values or α, we consider a mass M with model:

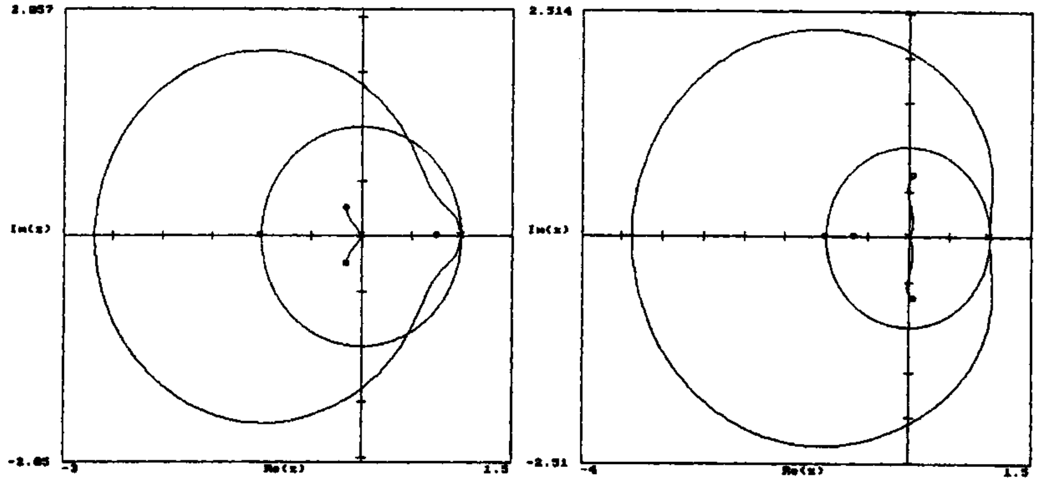
$$Z\left\{\frac{1}{Ms^2}\right\} = \frac{T^2}{2M} \frac{z + 1}{(z - 1)^2} \tag{21}$$

The elemental control algorithm consists on a fractional D or I action that results for $\alpha = 1/2$ and $\alpha = -1/2$, respectively. Figures 5, 6 and 7 show the root locus, in the z-domain, for controllers with actions $D^{1/2}$ and $I^{1/2}$, using the three methods referred.

Experiments with the different algorithms reveal that, for an high precision approximation, the series truncation method is superior and easy to develop. Therefore, this algorithm is adopted in the sequel. In order to investigate the robustness of the FDI-based control algorithms we introduced a nonlinear block in the forward path (Fig. 8).

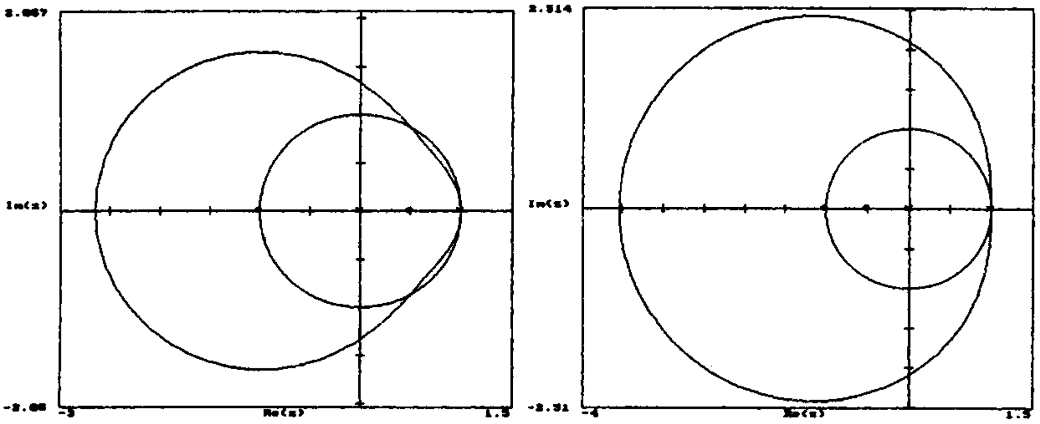
Four different phenomena in the actuator are considered for this system: saturation, deadzone, hysteresis and relay with the characteristics depicted in Figure 9. In all the cases, the parameters adopted in the experiments are $K = 10T^{1/2}$, $M = 1$ and a sample and hold time $T = 0.1$.

Figure 10 shows the linear system response (i.e. without the nonlinear block) for a unity step input and n-th order ($1 \leqslant n \leqslant 7$) approxi-



a) b)

FIGURE 5 Z-domain root locus for a mass and fractional control actions based on a 3-th order series approximation: a) $D^{1/2}(z) \approx K(1 - 1/2\,z^{-1} - 1/8\,z^{-2} - 1/16z^{-3})$ b) $I^{1/2}(z) \approx K(1 + 1/2\,z^{-1} + 3/8\,z^{-2} + 5/16z^{-3})$.



a) b)

FIGURE 6 Z-domain root locus for a mass and fractional control actions based on a linear interpolation: a) $D^{1/2}$ b) $I^{1/2}$.

mation to $D^{1/2}$ according with (9). It is clear that the higher the order of the approximation the better the response. Therefore, the *PD* like scheme, that is equivalent to the approximation of order $n = 1$, is inferior to the fractional order controller.

The robustness of the fractional algorithm over classical control actions is highlighted in the presence of a nonlinear phenomena.

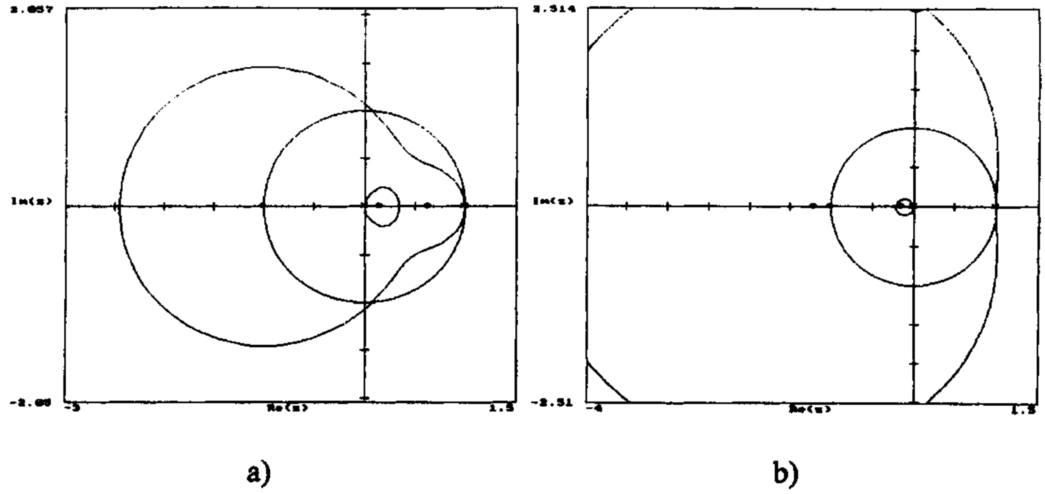


FIGURE 7 Z-domain root locus for a mass fractional control actions based on a quadratic interpolation: a) $D^{1/2}$ b) $I^{1/2}$.

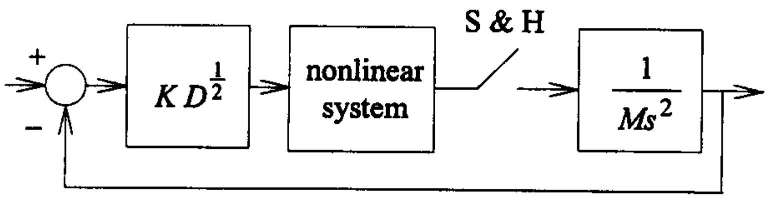


FIGURE 8 $D^{1/2}$ controller for a system with a mass and a nonlinear actuator.

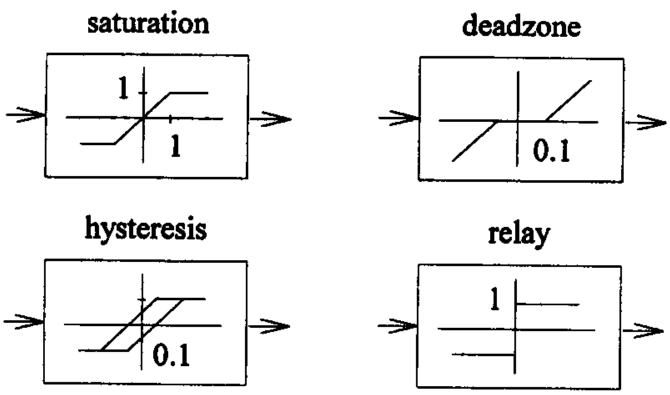


FIGURE 9 Nonlinear phenomena at the actuator: saturation, deadzone, hysteresis and relay.

Figure 11 shows that the response for a PD-like controller (i.e. the FDI controller for $n = 1$) is very sensitive to the saturation effect while the fractional controller remains stable.

In the same line of thought, Figures 12, 13 and 14 reveal that the 7-th order approximation to the $D^{1/2}$ controller is robust for a large

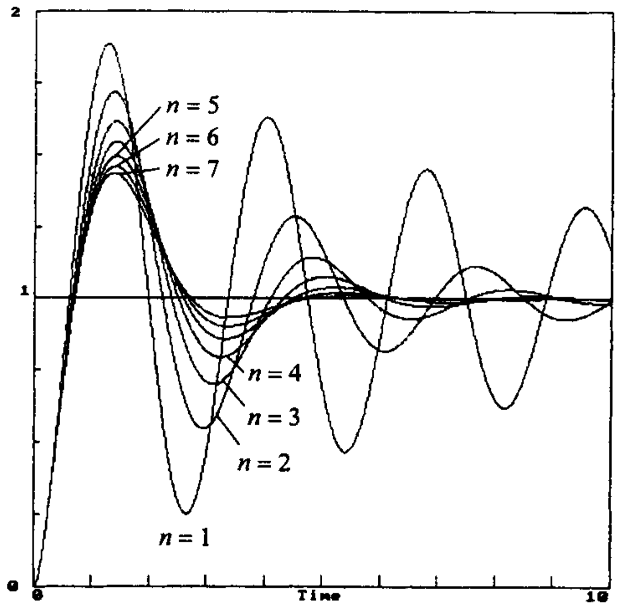


FIGURE 10 Linear system response (i.e. without the nonlinear block) for a n-th order series approximation to $D^{1/2}$.

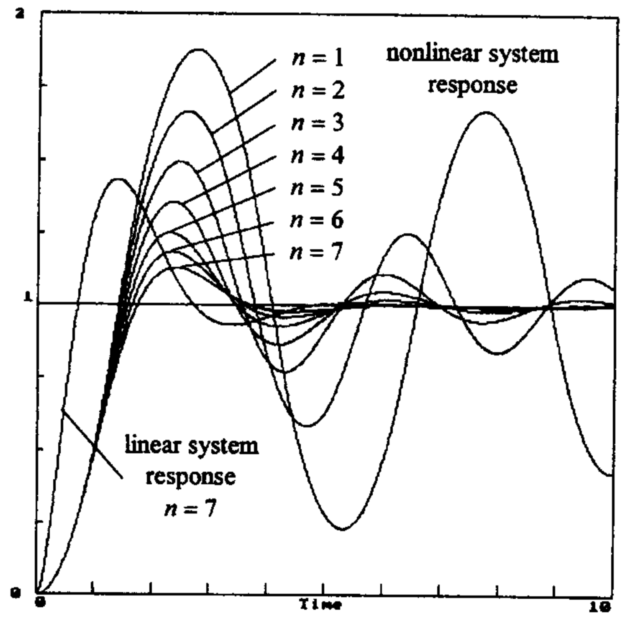


FIGURE 11 System response, with saturation for a n-th order series approximation to $D^{1/2}$.

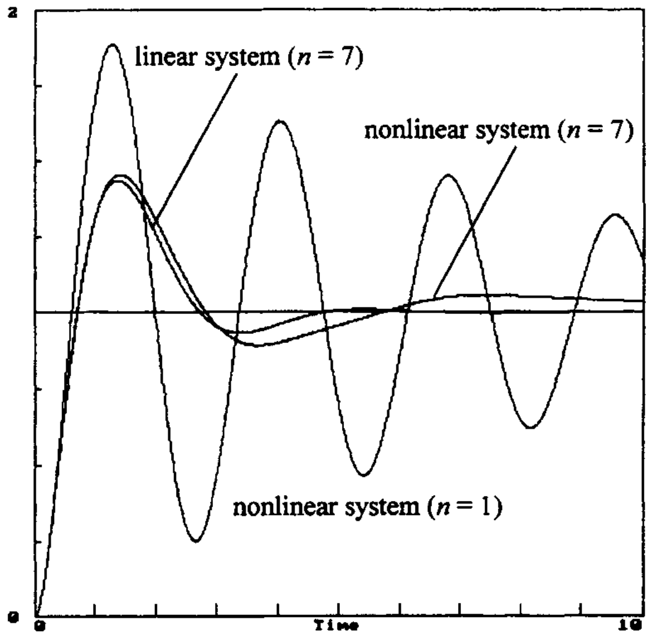


FIGURE 12 System response, with deadzone for a 7-th order series approximation to $D^{1/2}$.

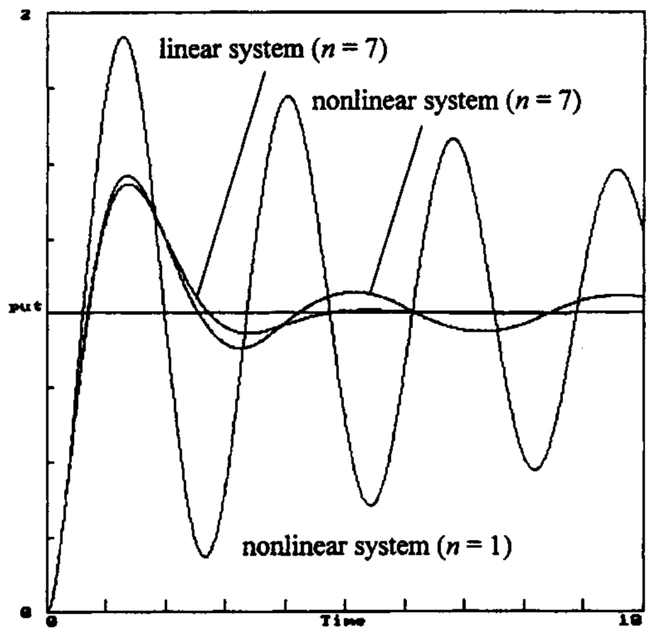


FIGURE 13 System response, with hysteresis for a 7-th order series approximation to $D^{1/2}$.

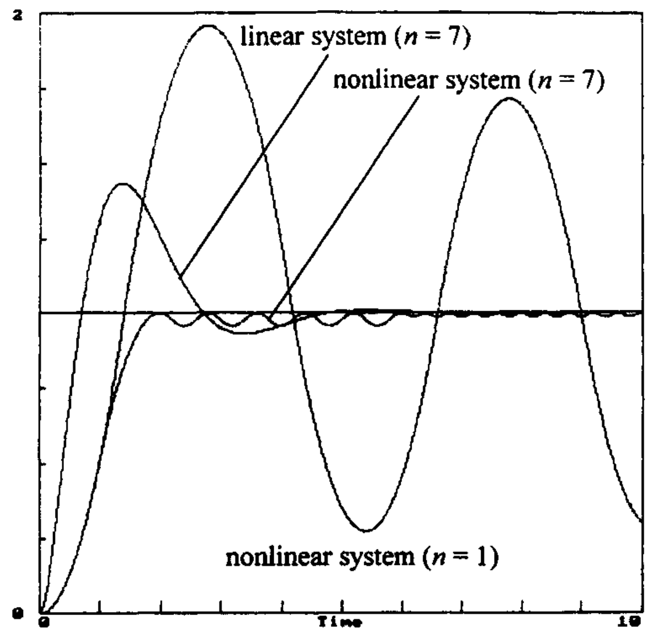


FIGURE 14 System response, with relay for a 7-th order series approximation to $D^{1/2}$.

range of nonlinear phenomena, having a better performance the higher the order of series adopted in expression (9).

6. CONCLUSIONS

The theory of FDI's is still in a research stage but the recent progress in the areas of chaos and fractal reveals promising aspects for future developments. In the field of automatic control systems some preliminary work has been carried out but the results are restricted to the frequency domain. In this paper a novel method for the FDI approximation was presented. The proposed algorithms adopt the time domain which makes them well suited for z-transform analysis and digital implementation. For a prototype mechanical system the control algorithm based on the new concepts reveal that classical P, I and D actions are special cases of a more broad paradigm. In fact, the FDI-based controllers reveal superior performances in the control of systems with nonlinear phenomena, being its limitations yet to be explored. In this line of thought, this study represents a first stage towards the development of motion control systems based on the theory of FDI's.

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