Fractional Dynamics: A Statistical Perspective

J. A. Tenreiro Machado
e-mail: jtm@isep.ipp.pt

Alexandra Galhano
e-mail: amf@isep.ipp.pt

Department of Electrotechnical Engineering,
Institute of Engineering of Porto,
Rua Dr. Antonio Bernardino de Almeida,
4200-072 Porto, Portugal

Fractional calculus is a mathematical paradigm that has been increasingly adopted to describe the dynamics of systems with hereditary characteristics, or that reflect an average of a large population of microelements. In this line of thought, this article analyzes the statistical dynamics of a system composed of a large number of micromechanical masses with backlash and impacts. We conclude that, while individual dynamics of each element has an integer-order nature, the global dynamics reveal the existence of both integer and fractional dynamics. [DOI: 10.1115/1.2833481]

1 Introduction

Fractional calculus (FC) is a branch of mathematics that deals with integrals and derivatives of noninteger order. FC is a misnomer, because it corresponds to the generalization of the operators to real or complex orders, and other more appropriate names can, and should, be adopted. Nevertheless, FC started with a discussion between Leibniz and L'Hospital (1695) about the calculation and the meaning of $D^{1/2}\{x^n\}$ and the denomination is used since then.

FC has been handled in pure mathematics [1–4] and its application in physics and engineering remained scarce until recently where its association with fractals and chaos motivated a renewed interest from the research community [5–14]. In spite of the developments, FC still has an exotic reputation, being often unclear its meaning and its usefulness from the viewpoint of applications.

Often, it is pointed out that one of the reasons for this state of affairs is the existence of several different interpretations for the fractional derivative [15–20]. In fact, we have several distinct perspectives for the fractional integral or derivative in opposition with the classical integer-order case where there is a unique, simple, geometrical interpretation. While this is viewed by some as a weakness we can, alternatively, observe that the real world reveals phenomena capable of being interpreted through several distinct paradigms, and that the history of the human research demonstrates that there are no "simple," "unique," and "definitive" explanations. Probably, the different interpretations of a fractional derivative have pros and cons, and adapt differently to the distinct physical phenomena, in such a way that the researcher can take advantage of the one that is more appropriate.

Fractional derivatives reveal that, inherently, the mathematical operator takes into account the memory of the past evolution of the variable, in contrast with the standard integer-order differential operator that considers only the local time history. This characteristic poses some difficulties, both for the initialization and the numerical calculation of the fractional operator but, on the other hand, constitutes a major advantage over the standard differential scheme because it leads to a concise description of the system dynamics.

Some interpretations of fractional derivatives are based on probabilistic concepts [19,20]. This perspective associated with the long memory of the fractional derivatives leads to the recent publication of studies addressing the modeling of some physical systems [21–28]. In simple terms, the main idea is that, in a dynamical system with many elemental components, each with a separate trajectory in the state space, the overall system dynamics is represented by the average of the individual contributions, either linear or nonlinear. If all elemental components exhibit a similar dynamics, then the average dynamics will be identical; however, in the presence of distinct individual behaviors, the global system model in average, will have a fractional dynamics.

Bearing these ideas in mind, this article analyzes the global dynamics of a system constituted by a large set of microscopic mechanical elements, each one consisting of two simple masses with backlash and impacts, while being driven by a white noise actuation force.

The rest of the article is organized as follows. Sections 2 and 3 present the system modeling, and the simulation and analysis, respectively. Finally, Sec. 4 outlines the main conclusions and points toward new experiments with the statistical evaluation of other dynamical systems.

2 System Modeling

In this section, a mechanical system (Fig. 1(*a*)) consisting of n identical components having, each one, an independent motion is considered. The overall driving force f_{total} is, therefore, distributed through the elemental components and the resulting motion x_{total} is the average of the individual displacements.

Each elemental component (Fig. 1(*b*)) consists of two masses M_1 and M_2, with displacements x_1 and x_2, respectively, having a backlash h, subjected to impacts [29,30] under the action of force f.

Individually, each mass follows classical Newton's law. Therefore, since it is considered that the force is driving M_2 without interaction, we have

$$0 = M_1\ddot{x}_1 \qquad (1a)$$

$$f = M_2\ddot{x}_2 \qquad (1b)$$

However, the existence of backlash establishes a dynamical interaction and the occurrence of impacts. Consequently, a collision between the masses M_1 and M_2 occurs when $x_1 = x_2 - \frac{1}{2}h$ or $x_1 = x_2 + \frac{1}{2}h$. The velocities of the masses M_1 and M_2 after the impact ($\dot{x}_1'$ and $\dot{x}_2'$) are related to their values before the impact ($\dot{x}_1$ and $\dot{x}_2$) through the expression

$$\dot{x}_1' - \dot{x}_2' = -\varepsilon(\dot{x}_1 - \dot{x}_2) \quad 0 \leqslant \varepsilon \leqslant 1 \qquad (2)$$

where ε is the coefficient of restitution that represents the dynamic phenomenon occurring in the masses during the impact. In the case of a fully plastic (inelastic) collision $\varepsilon=0$, while in the elastic case $\varepsilon=1$.

The principle of conservation of momentum requires that the momentum, immediately before and after the impact, must be identical:

$$M_1\dot{x}_1' + M_2\dot{x}_2' = M_1\dot{x}_1 + M_2\dot{x}_2 \qquad (3)$$

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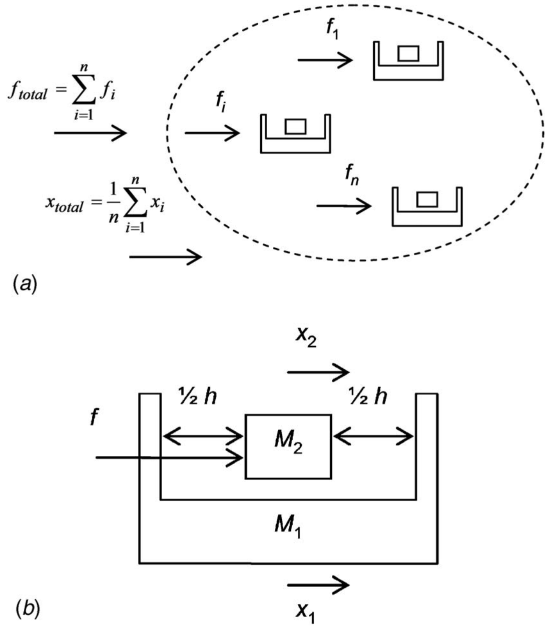

Fig. 1 Mechanical system; (*a*) multicomponent global structure and (*b*) elemental component, with two masses M_1 and M_2 with backlash h, subjected to impacts under the action of force f

From Eqs. (2) and (3), we can find the velocities of both masses after an impact, yielding

$$\dot{x}_1' = \frac{\dot{x}_1(M_1 - \varepsilon M_2) + \dot{x}_2(1+\varepsilon)M_2}{M_1 + M_2} \tag{4a}$$

$$\dot{x}_2' = \frac{\dot{x}_1(1+\varepsilon)M_1 + \dot{x}_2(M_2 - \varepsilon M_1)}{M_1 + M_2} \tag{4b}$$

The total kinetic energy loss E_L at an impact is determined by

$$E_L = \frac{1-\varepsilon^2}{2}\frac{M_1 M_2}{M_1 + M_2}(\dot{x}_1 - \dot{x}_2)^2 \tag{5}$$

3 System Simulation and Analysis

For the system simulation, it is considered that each elemental component is driven by a random force f in Eq. (1*b*) having a white noise spectra. In order to test the system linearity, different driving amplitudes are adopted according with the expression

$$f(t) = F_{\max}\mathrm{rnd}(t) \tag{6}$$

where t represents the time, $F_{\max}$ is the maximum amplitude, and $-1 \leq \mathrm{rnd}(t) \leq 1$ is a random number generator.

The motion of the masses M_1 and M_2, of each system microcomponent, is calculated through a Runge–Kutta four algorithm with fixed step dt [31], for a simulation time window T_w, and the corresponding velocities $\dot{x}_1$ and $\dot{x}_2$ are analyzed. It was verified that the numerical detection of the collision between the masses M_1 and M_2 is very sensitive to the integration step dt and, therefore, in order to minimize this problem during the simulations very small values were adopted. This phenomenon produces additional noise in the transfer functions, but can be viewed as a modeling effect that introduces some random characteristics in the backlash width h.

Figure 2 depicts a typical time history of $\dot{x}_1(t)$ and $\dot{x}_2(t)$ of one element, during one simulation $(i=1)$ with $T_w=10.0$ s, $F_{\max}=10.0$ N, $M_1=M_2=1.0$ kg, $h=10^{-2}$ m, $dt=5.0\times10^{-6}$ s. Figure 3 shows the corresponding transfer functions $H_1(j\omega)=F\{\dot{x}_1(t)\}/F\{f(t)\}$ and $H_2(j\omega)=F\{\dot{x}_2(t)\}/F\{f(t)\}$, where F repre-

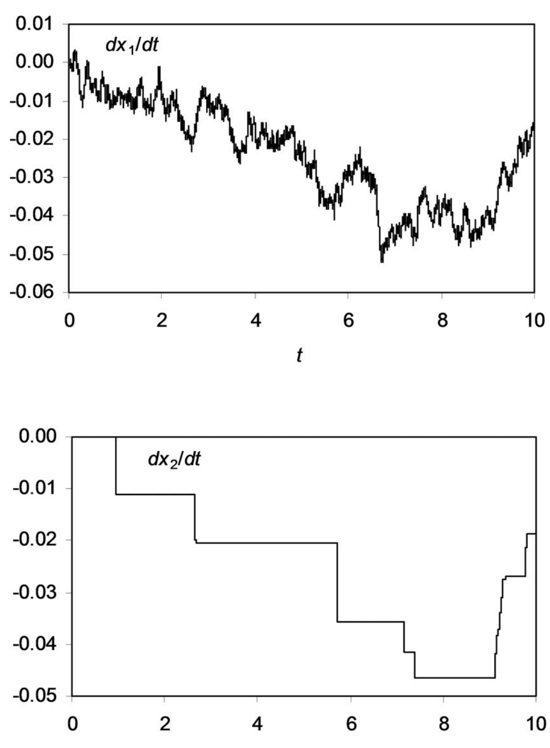

Fig. 2 Time history of $\dot{x}_1$ and $\dot{x}_2$ for one simulation with $T_w=10.0$ s, $F_{\max}=10.0$ N, $M_1=M_2=1.0$ kg, $h=10^{-2}$ m, $dt=5.0\times10^{-6}$ s

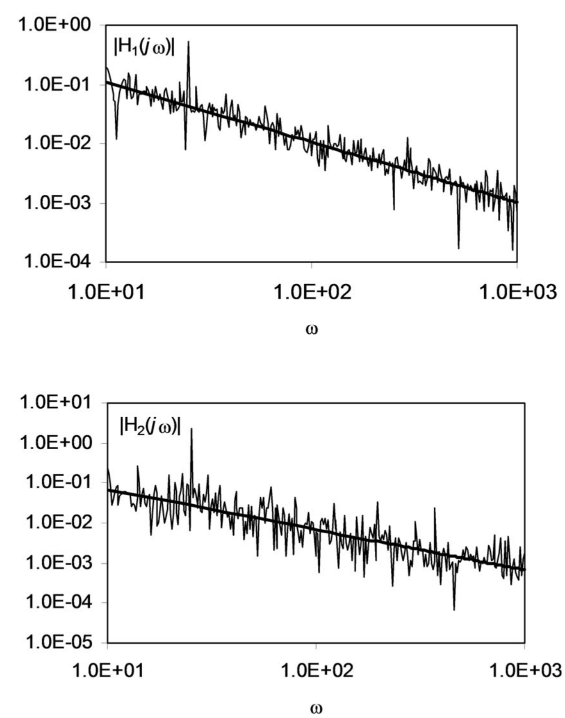

Fig. 3 Transfer functions $H_1(j\omega)$, $H_2(j\omega)$ and approximations $\{m_1^{-1},\alpha_1\}\approx\{1.18,-1.02\}$ and $\{m_2^{-1},\alpha_2\}\approx\{0.71,-1.01\}$, for one simulation with $T_w=10.0$ s, $F_{\max}=10.0$ N, $M_1=M_2=1.0$ kg, $h=10^{-2}$ m, $dt=5.0\times10^{-6}$ s

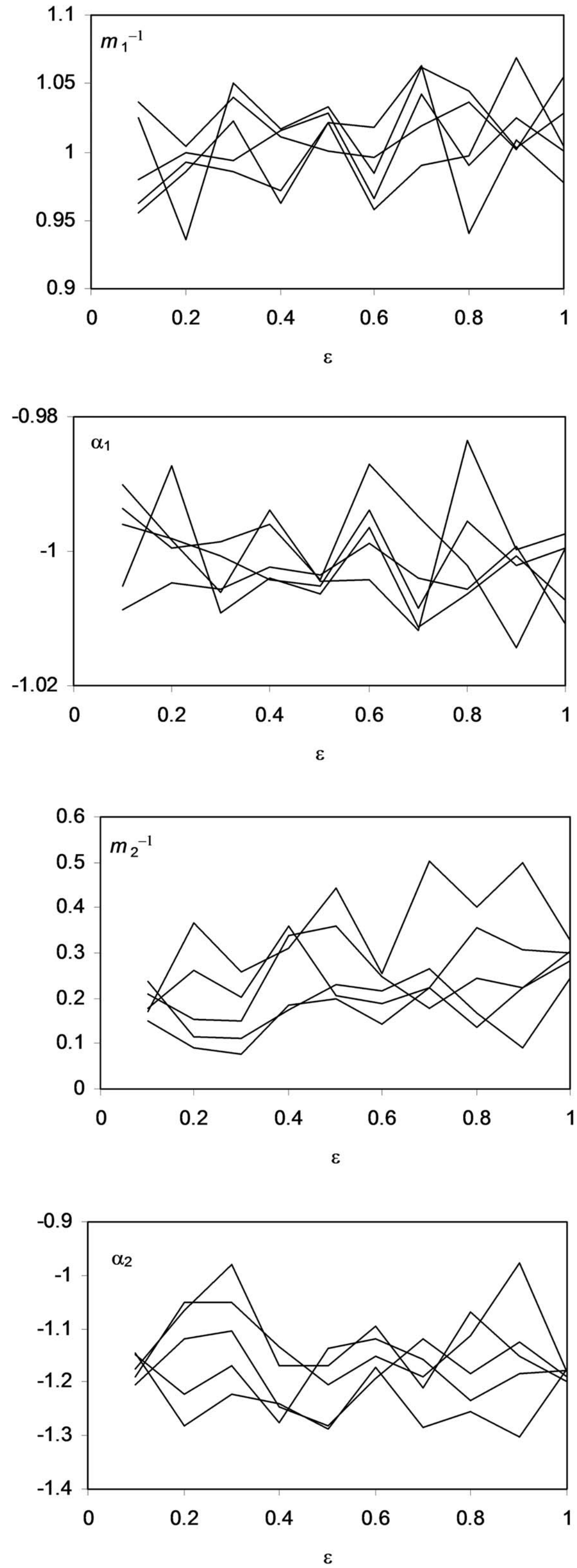

Fig. 4 Parameters of the average transfer functions versus ε for $F_{max}=\{10,20,30,40,50\}$, with $n=10^3$ microelements, $T_w=1.0$ s, $M_1=M_2=1.0$ kg, $h=10^{-2}$ m, $dt=5.0\times10^{-6}$ s

sents the Fourier transform, and $j=\sqrt{-1}$. It is clear that both transfer functions can be approximated by expressions of the type

$$H_k(j\omega) \approx m_k^{-1}\omega^{\alpha_k} \quad m_k, \alpha_k \in \Re \quad k=1,2 \tag{7}$$

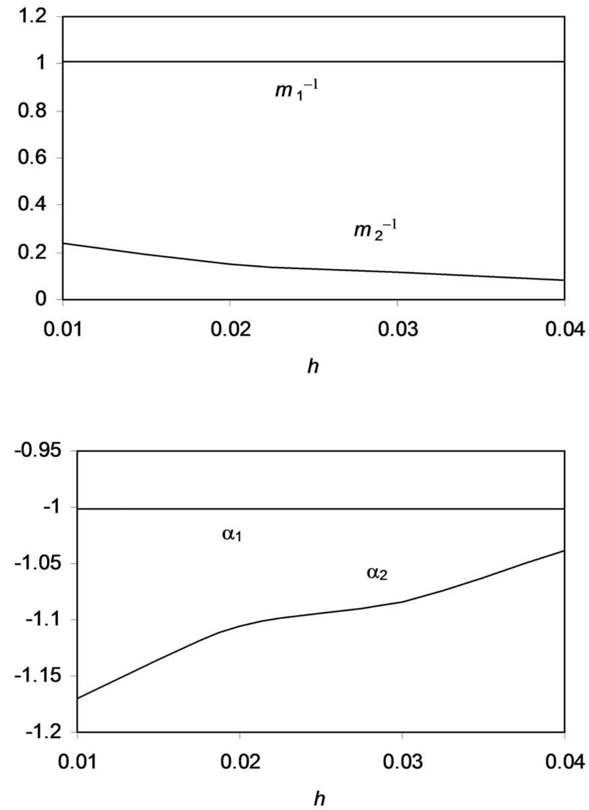

Fig. 5 Parameters $\{m_k^{-1},\alpha_k\}$, $k=1,2$, versus h, for $n=10^3$ microelements, $T_w=1.0$ s, $M_1=M_2=1.0$ kg, $h=10^{-2}$ m, $dt=5.0\times10^{-6}$ s

We get the approximations $\{m_1^{-1},\alpha_1\}\approx\{1.18,-1.02\}$ and $\{m_2^{-1},\alpha_2\}\approx\{0.71,-1.01\}$, which reveal an integer-order dynamics, in both cases, for the micromechanical element. However, we are mainly interested in the dynamic evaluation of the global system. Therefore, in order to evaluate the dynamics of the multicomponent system, a simulation with a sample of $n=10^3$ microelements is executed and the average of the n individual transfer functions is calculated.

Figure 4 shows the approximation parameters of the average transfer functions $[H_k(j\omega)]_{av}=\frac{1}{n}\sum_{i=1}^{n}[H_k(j\omega)]_i$, $k=1,2$, versus ε, for different levels of the exciting force $F_{max}=\{10,20,30,40,50\}$. We verify that the average transfer functions are smoother. Furthermore, we get $\{m_1^{-1},\alpha_1\}\approx\{1.01,-1.00\}$ and $\{m_2^{-1},\alpha_2\}\approx\{0.24,-1.17\}$, which is, clearly, an integer-order dynamics for M_1 but a fractional dynamics for M_2. The results are almost independent of the values of T_w, F_{max} (as long as the experiment assures a sufficient simulation time and an exciting force amplitude to produce M_2 motions larger than the backlash width), and ε but vary with the parameters M_1, M_2, and h. Moreover, the results for M_1 have considerable lower noise than those for M_2.

Bearing these facts in mind, a new set of experiments is developed to investigate the influence of the parameters upon the global dynamics. Figure 5 shows the variation of the parameters $\{m_k^{-1},\alpha_k\}, k=1,2$, versus the backlash width h. Since it was verified that the parameters are independent of $F_{max}=\{10,20,30,40,50\}$ and $\varepsilon=\{0.1,0.2,\cdots,0.9,1.0\}$, it is depicted the average value. This chart reveals, again, the integer and fractional dynamics of M_1 and M_2, respectively.

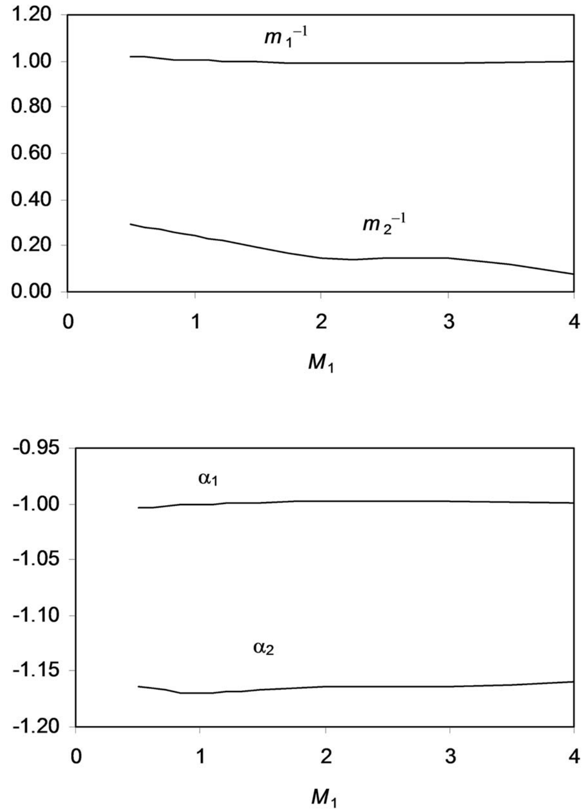

Fig. 6 Parameters $\{m_k^{-1}, \alpha_k\}$, $k=1,2$, versus M_1 for $M_2=1.0$ kg, $h=10^{-2}$ m, $n=10^3$ microelements, $T_w=1.0$ s, $dt=5.0\times10^{-6}$ s

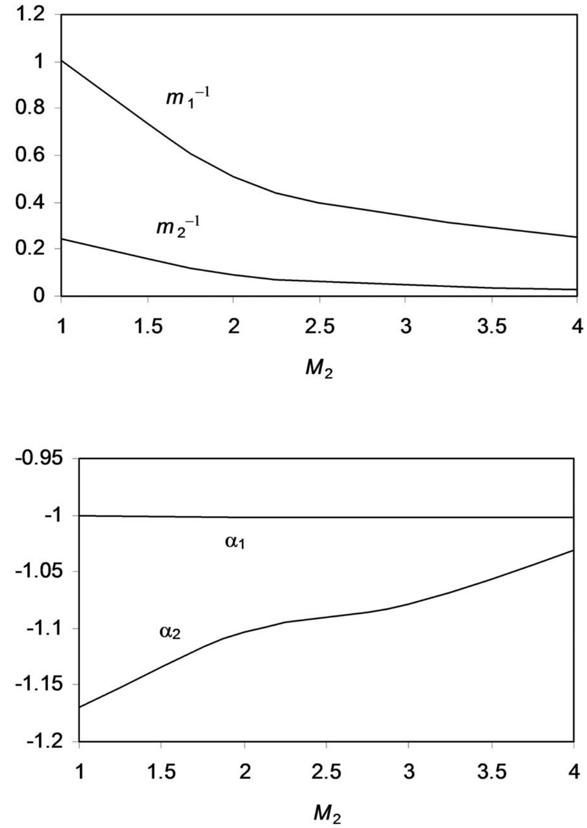

Fig. 7 Parameters $\{m_k^{-1}, \alpha_k\}$, $k=1,2$, versus M_2 for $M_1=1.0$ kg, $h=10^{-2}$ m, $n=10^3$ microelements, $T_w=1.0$ s, $dt=5.0\times10^{-6}$ s

Figures 6 and 7 depict the variation of the parameters $\{m_k^{-1}, \alpha_k\}, k=1,2$, with M_1 and M_2, respectively, for $h=10^{-2}$ m. We verify that the parameters $m_k^{-1}, k=1,2$, have a small variation with M_1, but are very sensitive to M_2. In fact, we have the power law dependence $m_1^{-1} \approx M_1^{-0.01}$ and $m_2^{-1} \approx 0.22 M_2^{-0.57}$ for $M_2 = 1.0$ kg, while we get $m_1^{-1} \approx M_1^{-0.99}$ and $m_2^{-1} \approx 0.25 M_2^{-1.58}$ for $M_1 = 1.0$ kg. In both cases, it yields $\alpha_1 \approx 1.0$, close to an integer value, while α_2 is clearly fractional and sensitive to the values of the colliding masses.

The complete understanding of the phenomena requires the cross checking with other values of h, ε, M_1, and M_2. Due to the computational burden for the simulations, and the multiplicity of combinations of numerical values, such experiments are presently ongoing. Another aspect that deserves further study is the global system architecture. In fact, while adopted an average of the micro mechanical individual (and independent) dynamics was adopted, many other configurations can be envisaged and their influence upon the global resulting dynamics needs to be studied.

4 Conclusions

FC is a mathematical paradigm that has been increasingly adopted to describe the dynamics of systems with hereditary characteristics, or that reflect an average of a large population of micro elements. In this line of thought, this article analyzed the statistical dynamics of a system composed of a large number of micromechanical masses with backlash and impacts.

We conclude that, while individual dynamics of each element has an integer-order nature, the global dynamics reveal the existence of both integer and fractional dynamics. The dependence of the system numerical parameters and the influence of the global system architecture upon the global dynamics are matters that need further research.

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