Analysis of the Van der Pol Oscillator Containing Derivatives of Fractional Order

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Abstract: In this paper a modified version of the classical Van der Pol oscillator is proposed, introducing fractional-order time derivatives into the state-space model. The resulting fractional-order Van der Pol oscillator is analyzed in the time and frequency domains, using phase portraits, spectral analysis and bifurcation diagrams. The fractional-order dynamics is illustrated through numerical simulations of the proposed schemes using approximations to fractional-order operators. Finally, the analysis is extended to the forced Van der Pol oscillator.

Key words: Fractional calculus, Van der Pol oscillator, nonlinear oscillators, fractional order operators

1. INTRODUCTION

The study of nonlinear oscillators has been important in the development of the theory of dynamical systems. The Van der Pol oscillator (VPO), described by a second-order nonlinear differential equation, can be regarded as describing a mass–spring–damper system with a nonlinear position-dependent damping coefficient or, equivalently, an RLC electrical circuit with a negative-nonlinear resistor, and has been used to develop models in many applications, such as electronics, biology or acoustics. It represents a nonlinear system with an interesting behaviour that arises naturally in several applications.

This kind of nonlinear oscillator was used by Van der Pol in the 1920s to study oscillations in vacuum tube circuits (part of the early radios). In standard form, it is given by a second-order nonlinear differential equation of type:

$$\ddot{x} + \alpha\left(x^2 - 1\right)\dot{x} + x = 0 \tag{1}$$

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Figures 1–7, 9–11 appear in color online: http://jvc.sagepub.com

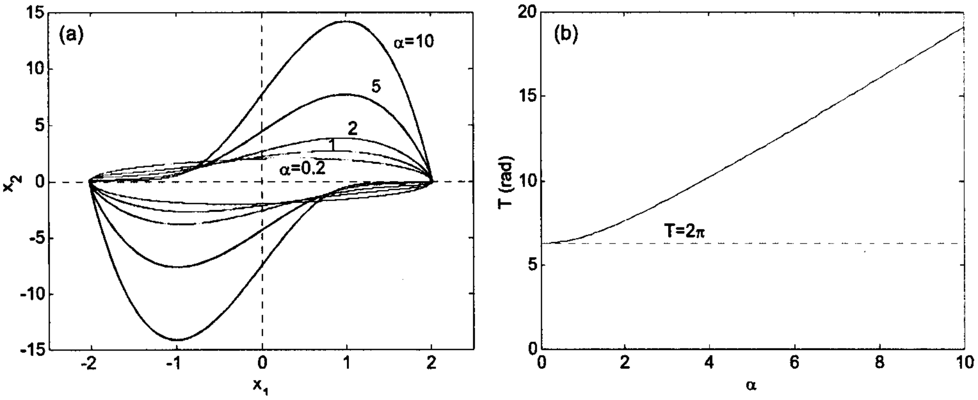

Figure 1. Van der Pol oscillator: (a) phase portraits; (b) period of oscillation $T = 2\pi/\omega$ versus the parameter α.

where α is a control parameter that reflects the degree of nonlinearity of the system. The equation (1) possesses a periodic solution that attracts other solution except the trivial one at the unique equilibrium point $x = \dot{x} = 0$.

The state-space model of the system, with $x_1 = x$, $x_2 = \dot{x}$ is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\alpha\left(x_1^2 - 1\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2}$$

Figure 1(a) shows the phase portraits of the Van der Pol equation (2) for initial conditions $x_1(0) = 0$, $x_2(0) = 2$ as the control parameter α is varied.

Clearly, the phase portraits depend on α, namely: $\alpha = 0$, harmonic oscillator; $\alpha > 0$, stable limit cycle; α increasing, nonlinearity increasing. The amplitude of oscillations is nearly constant at the value 2, but the frequency of oscillation ω (period $T = 2\pi/\omega$) depends on α, as shown in Figure 1(b). For lower values of α the frequency is approximately $\omega =$ 1 rad/s ($T = 2\pi$ rad).

In this paper we investigate the influence of a fractional-order time derivative introduced in the Van der Pol equation dynamics (2). The modified equation is called the fractional Van der Pol oscillator. The system is analysed both in time and frequency domains and its dynamics illustrated through phase portraits, frequency spectra and bifurcation diagrams. The forced version of the system is also considered.

Bearing these ideas in mind, the article is organized as follows. Section 2 reviews the fundamentals of fractional calculus. Section 3 presents a frequency approximation method of fractional-order integrators. This type of approximations is used in the simulation of the fractional VPO system. In Section 4 we propose several versions of the VPO containing fractional derivatives. It is also presented some numerical simulations of the fractional VPO system under study. In Section 5 we consider the forced version of the fractional VPO system. Finally, Section 6 draws the main conclusions.

2. FUNDAMENTALS OF FRACTIONAL CALCULUS

The fractional calculus concerns the study and applications of integrals and derivatives of arbitrary order (real or complex order). There are different approaches to the fractional calculus, not all being equivalent. The two most commonly used definitions are the Riemann–Liouville and the Grünwald–Letnikov definitions (Oldham and Spanier 1974; Podlubny 1999; Hilfer 2000; Oustaloup 1995). The Riemann–Liouville definition of the fractional-order derivative is ($\lambda > 0$):

$$ {}_aD_t^{\lambda} f(t) = \frac{1}{\Gamma(n-\lambda)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\lambda-n+1}} d\tau, \quad n-1 < \lambda < n \tag{3} $$

where $\Gamma(x)$ is the well-known Gamma function in x.

On the other hand, the Grünwald–Letnikov definition is formulated as ($\lambda \in \Re$):

$$ {}_aD_t^{\lambda} f(t) = \lim_{h \to 0} \frac{1}{h^{\lambda}} \sum_{k=0}^{\left[\frac{t-a}{h}\right]} (-1)^k \binom{\lambda}{k} f(t-kh) \tag{4} $$

where h is the time increment and $[x]$ means the integer part of x.

For a wide class of functions, important for applications, both definitions are equivalent (Podlubny 1999). This allows one to use the Riemann–Liouville definition during problem formulation, and then turn to the Grünwald–Letnikov definition to obtain a numerical solution.

An alternative definition, which is useful for the analysis and control design of dynamic systems, is given by the Laplace transform (L) method. Considering vanishing initial conditions, this definition is given by the expression ($\lambda \in \Re$):

$$ L\left\{ {}_aD_t^{\lambda} f(t) \right\} = s^{\lambda} F(s) \tag{5} $$

where $F(s) = L\{f(t)\}$. Expression (5) is a direct generalization of the integer-order scheme with the multiplication of the signal transform $F(s)$ by the Laplace s-variable raised to a fractional value λ. The frequency response of (5) is represented in the magnitude Bode diagram by a straight line of slope 20λ dB/decade and in the phase Bode diagram by a horizontal line positioned at $\lambda\pi/2$ rad.

3. APPROXIMATIONS OF FRACTIONAL-ORDER OPERATORS

From expressions (3)–(5) we note that the fractional-order operator has an unlimited memory, the integer-order operators being particular cases of this general case in which the memory is limited. These operators are characterized by having irrational continuous transfer functions in the Laplace domain or infinite-dimensional discrete transfer functions in the time domain. This fact poses evaluation problems when used in simulations. Then, the usual approach for analysing fractional-order systems is the development of continuous and discrete integer-order approximations of fractional-order operators (Oustaloup 1995; Machado 2001; Vinagre et al. 2000; Hwang et al. 2002).

In this paper we use Charef's approximation frequency method (Charef et al. 1992) to obtain rational-type approximations of the fractional-order integrator $1/s^{\lambda}$. The basic idea is to approximate the slope of the magnitude Bode diagram of the transfer function of a single-fractional power pole of the form:

$$\frac{1}{s^{\lambda}} \approx \frac{1}{\left(1+\dfrac{s}{p_T}\right)^{\lambda}} \tag{6}$$

with a succession of zeros and poles leading to straight lines with alternate slopes of 0 dB/decade and –20 dB/decade, and whose combination provides the desired slope of -20λ dB/decade over the required range of frequency. Thus, the approximation is given by:

$$H(s) = \frac{\prod_{i=0}^{N-1}\left(1+\dfrac{s}{z_i}\right)}{\prod_{i=0}^{N}\left(1+\dfrac{s}{p_i}\right)} \tag{7}$$

where the coefficients are computed to obtain maximum deviation from the original magnitude response in the frequency domain of y dB. Defining:

$$a = 10^{y/10(1-\lambda)}, \quad b = 10^{y/10\lambda}, \quad ab = 10^{y/10\lambda(1-\lambda)} \tag{8}$$

the poles and zeros of the approximation (7) are obtained by applying the following formulae:

$$p_0 = p_T\sqrt{b}, \quad p_i = p_0\,(ab)^i\,, \quad z_i = a p_0\,(ab)^i\,. \tag{9}$$

The number of poles and zeros is related to the desired bandwidth and the error criteria used by the expression:

$$N = \left[\frac{\log\left(\dfrac{\omega_{\max}}{p_0}\right)}{\log(ab)}\right] + 1. \tag{10}$$

4. VAN DER POL OSCILLATOR WITH FRACTIONAL DERIVATIVES

The standard Van der Pol equation (1) is modelled by a differential equation for which the elastic restoring force is a linear function of the dependence variable. However, it may be of interest to consider modifications to this equation in which the dependent variable x and/or its derivatives occur to some fractional power (Arena et al. 2000; Hartley et al. 1995; Ahmad and Sprott 2003; Mickens 2002, 2003; Pereira et al. 2004; Barbosa et al. 2004). Such nonlinear differential equations are usually called fractional Van der Pol equations.

Mickens (2002, 2003) investigated the following two equations:

$$\ddot{x} + \alpha\left(x^2 - 1\right)\dot{x} + x^{1/3} = 0 \tag{11}$$

$$\ddot{x} + \alpha\left(x^2 - 1\right)(\dot{x})^{1/3} + x = 0. \tag{12}$$

More recently, Pereira et al. (2004) considered a fractional version of the Van der Pol equation given by:

$$x^{\lambda} + \alpha\left(x^2 - 1\right)\dot{x} + x = 0, \quad 1 < \lambda < 2 \tag{13}$$

$$\begin{bmatrix} \dot{x}_1 \\ x_2^{(\lambda)} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\alpha\left(x_1^2 - 1\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{14}$$

which is obtained by substituting the capacitance by a fractance in the nonlinear RLC circuit model. Barbosa et al. (2004) also suggested the introduction of a fractional-order time derivative in the state-space equations (2) of the standard VPO in the form:

$$\begin{bmatrix} x_1^{(\lambda)} \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\alpha\left(x_1^2 - 1\right) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{15}$$

where $0 < \lambda < 1$ and $\alpha > 0$. A similar approach was performed for the Duffing (Arena et al. 2000) and Chua (Hartley et al. 1995) systems. Note that the system (15) reduces to the classical VPO (2) when $\lambda = 1$ and that the total system order is changed to $\lambda + 1 < 2$. The differential equation of system (15) is given by:

$$x^{(1+\lambda)} + \alpha\left(x^2 - 1\right)x^{(\lambda)} + x = 0, \quad 0 < \lambda < 1 \tag{16}$$

In this article we investigate equation (16).

The block diagram representation of system (16) is illustrated in Figure 2. The fractional-order integrator $1/s^{\lambda}$ $(0 < \lambda < 1)$ was simulated by using approximations of type (7) with $p_T = 0.01$, $\omega_{\max} = 100$ rad/s and $y = 2$ dB. Figure 3 shows the phase portraits for initial conditions $x_1(0) = 0$ and $x_2(0) = 1$ as the fractional-order λ (left plot) and the control parameter α (right plot) are varied, respectively. In both cases, we verify significant variations of the limit cycle, revealing a large impact of the λ-order derivative upon system dynamics.

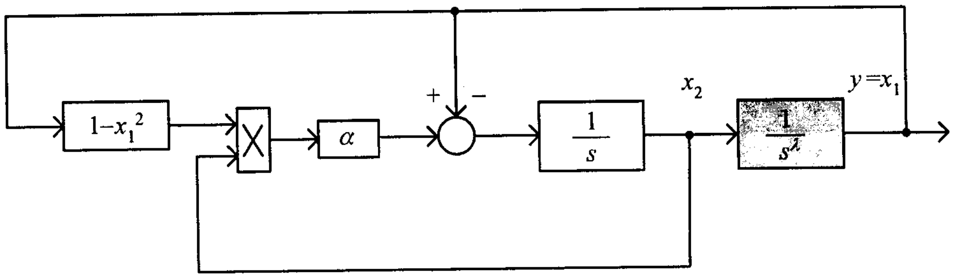

Figure 2. Block diagram of the fractional Van der Pol system under study.

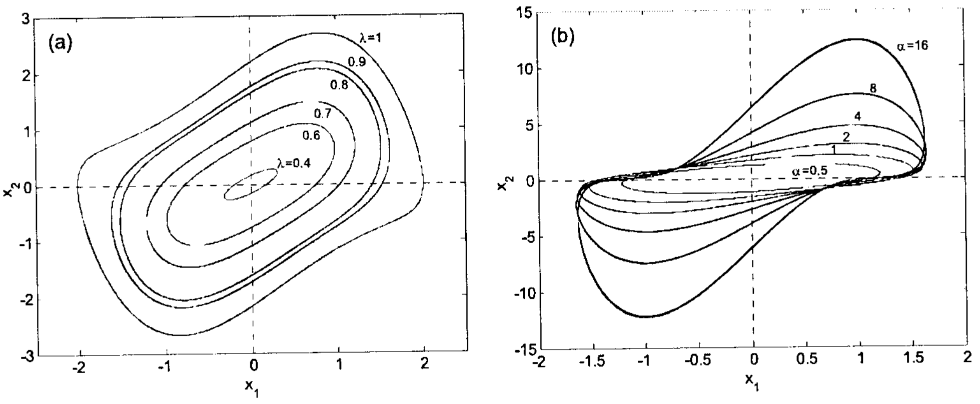

Figure 3. Phase portraits: (a) $\lambda = \{0.4, 0.6, 0.7, 0.8, 0.9, 1.0\}$ and $\alpha = 1$; (b) $\lambda = 0.8$ and $\alpha = \{0.5, 1, 2, 4, 8, 16\}$.

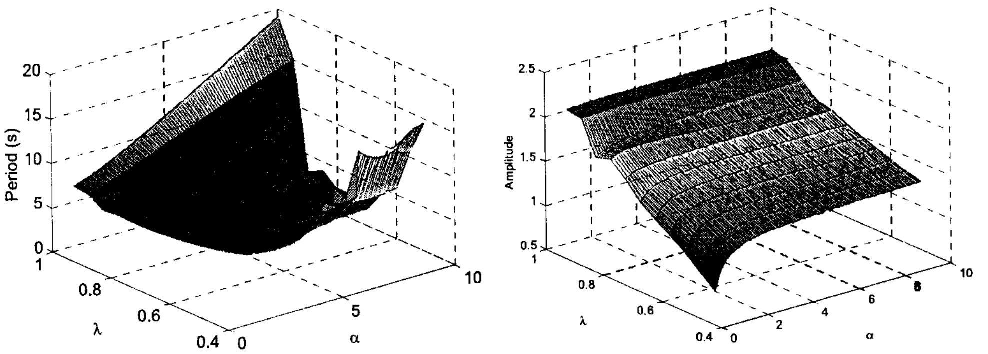

Figure 4. Limit cycle: period and amplitude of the output oscillation for $1 \leq \alpha \leq 10$ and $0.5 \leq \lambda \leq 1$.

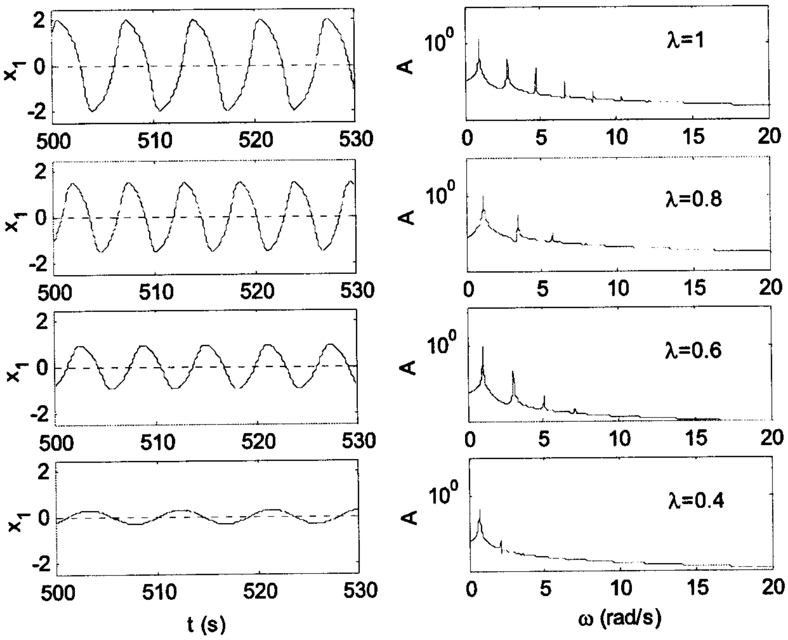

Figure 5. Time responses and Fourier spectra for $\lambda = \{1, 0.8, 0.6, 0.4\}$ and $\alpha = 1$.

In order to clarify this point, Figure 4 illustrates the amplitude and the period of the output oscillation. It is clearly seen the large variation of the limit cycle, particularly in the period of the oscillation.

Figures 5 and 6 show the steady-state time responses and the amplitude Fourier spectra of the output $x_1(t)$ for several values of λ and for $\alpha = 1$ and $\alpha = 5$, respectively. The frequency spectrum was evaluated by using the FFT (fast Fourier transform) over $N = 2^{15}$ points after elapsing the initial transient up to 100 s of the signal output $x_1(t)$. Once more, we observe the variation of the limit cycle as function of λ, noting that the amplitude gets smaller as λ is decreased. On the other hand, analysing the Fourier spectra, we verify that the multiplicity of peaks and the amplitude of these peaks varies with α, which is in accordance with the time responses. Also note that the energy of the output signal, is not only concentrated in the peaks (fundamental and integer-odd harmonics), but distributed along all frequency domain (Barbosa et al. 2004). This fact is characteristic of chaotic systems showing that the fractional VPO system presents chaotic limit cycles, like in the case of the classical VPO (Mahmoud and Farghaly 2004).

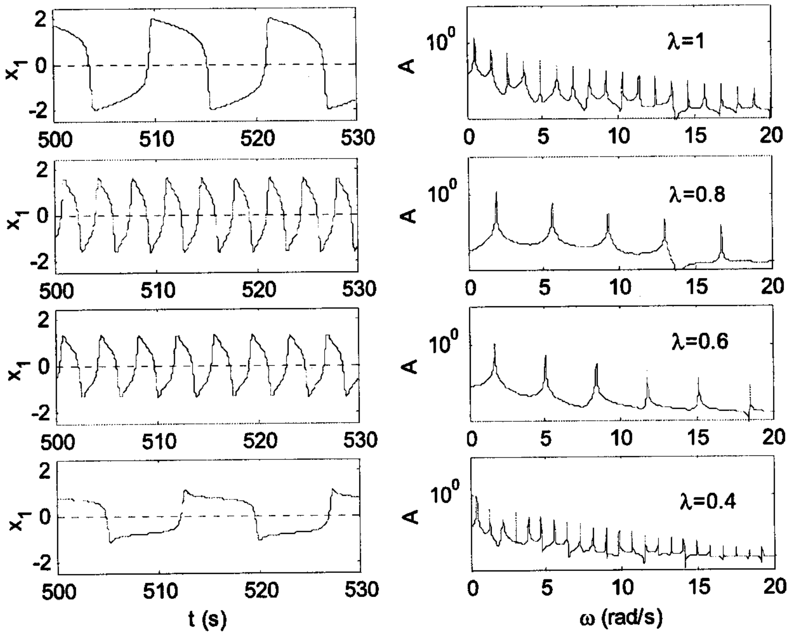

Figure 6. Time responses and Fourier spectra for $\lambda = \{1, 0.8, 0.6, 0.4\}$ and $\alpha = 5$.

5. FORCED VAN DER POL OSCILLATOR WITH FRACTIONAL DERIVATIVES

Let us now consider the forced version of the fractional VPO system defined in state-space form as:

$$\begin{aligned} x_1^{(\lambda)} &= x_2 \\ \dot{x}_2 &= -x_1 - \alpha\left(x_1^2 - 1\right)x_2 + f\cos\left(\omega_f t\right) \end{aligned} \tag{17}$$

where f and ω_f are the amplitude and the frequency of the forcing sinusoidal input, respectively. The block diagram representation of system (17) is depicted in Figure 7.

It is well known that for the parameters $\alpha = 5$, $\omega_f = 2.46$ rad/s and $\lambda = 1$ the classical forced VPO exhibits chaos. For the forced fractional VPO, by modifying the order λ, the system will now reveal a different behaviour. For example, Figure 8 shows the bifurcation diagram of the sampled output position $x_1(nT)$ as a function of the forcing amplitude f for a fractional-order $\lambda = 0.85$. This graph was obtained by applying the method of Poincaré sections.

From the bifurcation plot we can distinguish different modes of the forced fractional VPO system, namely: periodic motion, quasiperiodic motion and period locked motion. These types of motion are illustrated in Figures 9–11 by phase portraits and Fourier spec-

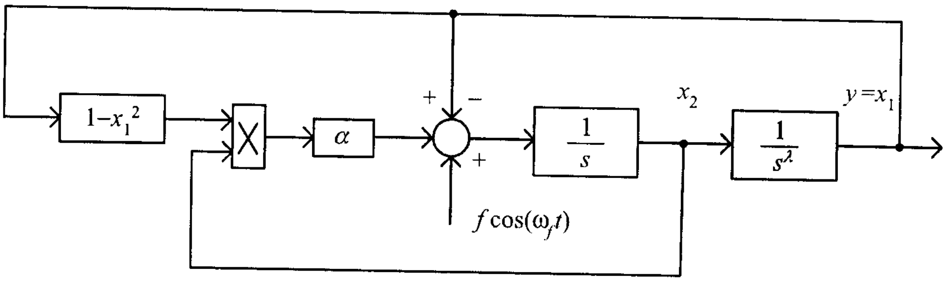

Figure 7. Block diagram of the forced fractional Van der Pol system under study.

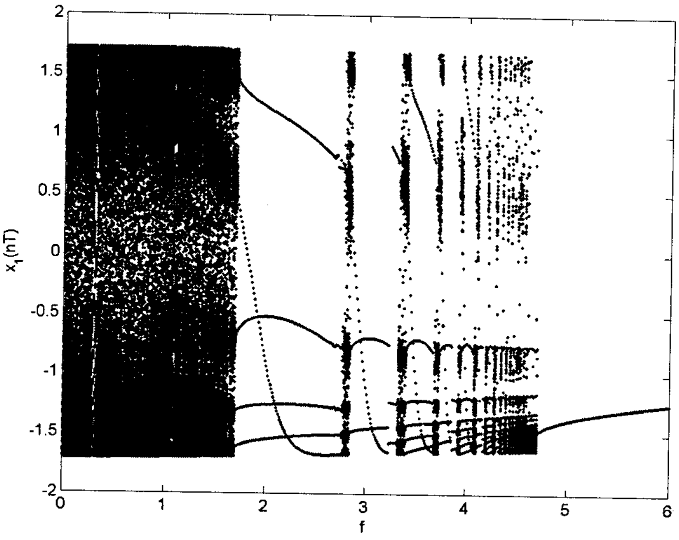

Figure 8. Bifurcation diagram for $\alpha = 5$, $\omega_f = 2.46$ rad/s and fractional-order $\lambda = 0.85$ versus the forcing amplitude *f*.

tra. In the periodic motion the phase-plane exhibits period doubling, as shown in Figure 9. Figure 10 depicts the quasiperiodic motion in which the system is oscillating at multiple or sub-multiple periods of the forcing frequency. In this case the frequency and amplitude vary with time. Finally, Figure 11 illustrates the period-locked motion in which the system is oscillating at the forcing frequency. Note that all these modes correspond to a periodic behaviour of the system. The non-periodic behaviour is characterized by chaos (or sensitivity to initial conditions). It is well known that the classical forced Van der Pol equation can display chaos for specific sets of parameters, not always easy to find. The same difficulty can be expected for the case of the forced fractional Van der Pol equation.

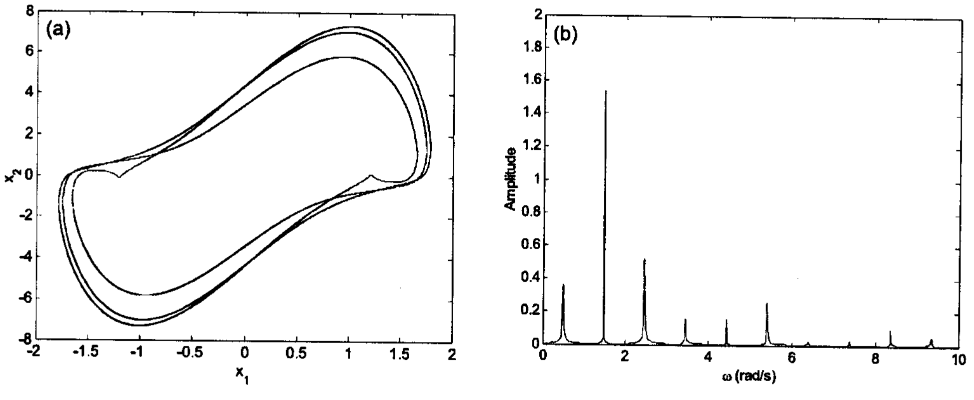

Figure 9. Phase plane (a) and Fourier spectrum (b) for $\alpha = 5$, $\omega_f = 2.46$ rad/s, $f = 2.0$ and fractional-order $\lambda = 0.85$: periodic motion.

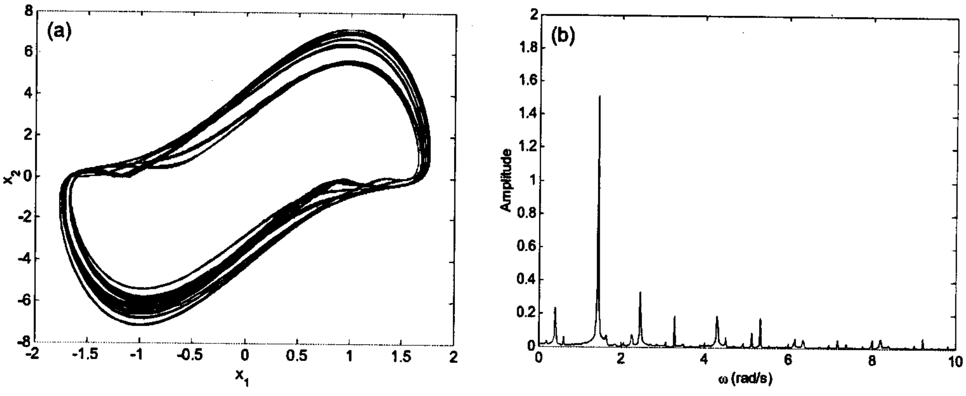

Figure 10. Phase plane (a) and Fourier spectrum (b) for $\alpha = 5$, $\omega_f = 2.46$ rad/s, $f = 1.5$ and fractional-order $\lambda = 0.85$: quasiperiodic motion.

6. CONCLUSIONS

In this paper we have proposed several versions of the modified Van der Pol equation. Such modifications consisted in the introduction of a fractional-order time derivative in the sate-space equations of the standard Van der Pol oscillator. The unforced and forced versions of the resulting fractional-order Van der Pol oscillators were studied in the time and frequency domains. The results reveal that fractional-order systems can exhibit different behaviour from those obtained with the standard Van der Pol oscillator, depending on the order's derivative (or the system order). The fractional-order can act as a modulation parameter that may be useful for a better understanding and control of such systems.

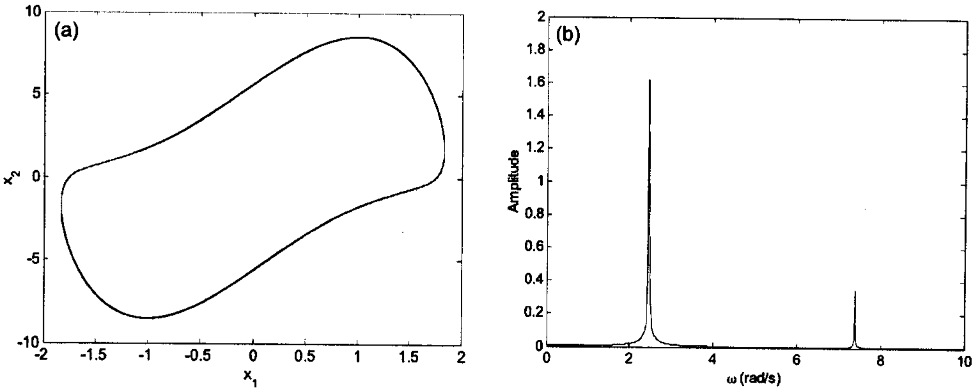

Figure 11. Phase plane (a) and Fourier spectrum (b) for $\alpha = 5$, $\omega_f = 2.46$ rad/s, $f = 5.5$ and fractional-order $\lambda = 0.85$: period locked motion.

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