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A PROBABILISTIC INTERPRETATION OF THE FRACTIONAL-ORDER DIFFERENTIATION

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Abstract

The theory of Fractional Calculus (FC) is a useful mathematical tool for applied sciences. Nevertheless, FC is somehow hard to tackle and only in the last decades researchers were motivated for the application of the associated concepts. There are several reasons for this state of affairs, namely the apparent 'sufficiency' of classical differential calculus for real-world applications, the plethora of different definitions for fractional derivatives and integrals and the lack of a simple interpretation for such formulae. In what concerns the FC usefulness in the case of physics and engineering sciences, the progress in the areas of chaos and fractals lead to the development of fractional-order models and algorithms. On the other hand, the conceptual analysis of a fractional integral or a fractional derivative has also been addressed, but a simple interpretation is not yet completely established.

This paper discusses a probabilistic interpretation of the fractional-order derivative, based on the Grünwald-Letnikov definition, that reduces to the standard geometric interpretation for the limit cases of integer order, namely for the derivatives of order one and zero.

Mathematics Subject Classification: 26A33, 26A42, 83C99, 44A35, 45D05

Key Words and Phrases: fractional derivative, fractional calculus, geometric interpretation, probabilistic interpretation

1. Introduction

Fractional calculus (FC) goes back to the beginning of the theory of differential calculus. FC deals with the generalization of standard integrals and derivatives to a non-integer, or even complex, order [1-5]. Therefore, in this line of thought a wide range of potential fields of application are possible, by bringing to a broader paradigm concepts of physics, chemistry and engineering [6-23]. Nevertheless, until recently, FC was an 'unknown' mathematical tool for the applied sciences, being present day interest mainly due to the developments in the areas of chaos dynamics and fractals [25-27].

One of the reasons for this state of affairs is the lack of a simple interpretation for a fractional order derivative. In fact, while for the integer-order case we have a common geometric concept, in the fractional-order case we have problems in finding a clear and comprehensive reasoning scheme. Furthermore, the existence of several alternative definitions for an integral or a derivative of non-integer order, leads to an extra difficulty in capturing the "adequate" point of view.

Having these ideas in mind, several researchers proposed different approaches for the interpretation of fractional-order integrals and derivatives [27-40], but the fact is that a final paradigm is not yet well established. This paper presents an alternative point of view based on the probability theory that reduces to the standard interpretation for the case of a derivative of integer order.

2. A probabilistic perspective of the fractional-order derivative

In this paper it is addressed the Grünwald-Letnikov definition of a derivative of fractional order α of the signal $x(t)$, $D^\alpha[x(t)]$, given by the expression:

$$D^\alpha\left[x\left(t\right)\right] = \lim_{h\to 0}\left[\frac{1}{h^\alpha}\sum_{k=0}^{\infty}\gamma\left(\alpha,k\right)x\left(t-kh\right)\right] \tag{2.1}$$

$$\gamma\left(\alpha,k\right) = \left(-1\right)^k\frac{\Gamma\left(\alpha+1\right)}{k!\Gamma\left(\alpha-k+1\right)} \tag{2.2}$$

where Γ is the gamma function and h is the time increment.

Analyzing (2.2) we see that, for $0 < \alpha < 1$, we have:

$$\gamma(\alpha,0) = 1 \tag{2.3}$$

$$-\sum_{k=1}^{\infty}\gamma(\alpha,k)=1 \tag{2.4}$$

From the point of view of probability theory these results lead directly to the following conclusions:

- According with (2.3) the "present" (*i.e.* $x(0)$) is seen in expression (2.1) with probability one;

- Due to (2.4) the totality of the "past/future" (*i.e.* $x(-h), x(-2h),\ldots$) is also captured with probability one; however, each sample of $x(t)$ is weighted with a given probability, that is higher the closer we are to the "present".

Consequently, expression $-\sum_{k=1}^{\infty}\gamma(\alpha,k)\,x(t-kh)$ can be viewed as the expected value of the random variable X, $E(\mathrm{X})$, such that $P(\mathrm{X}=x(kh))=|\gamma(\alpha,k)|$, $k=1, 2,\ldots, 0<\alpha<1$.

Bearing these facts in mind, Figure 1 shows the geometric interpretation of (1) in the probabilistic perspective.

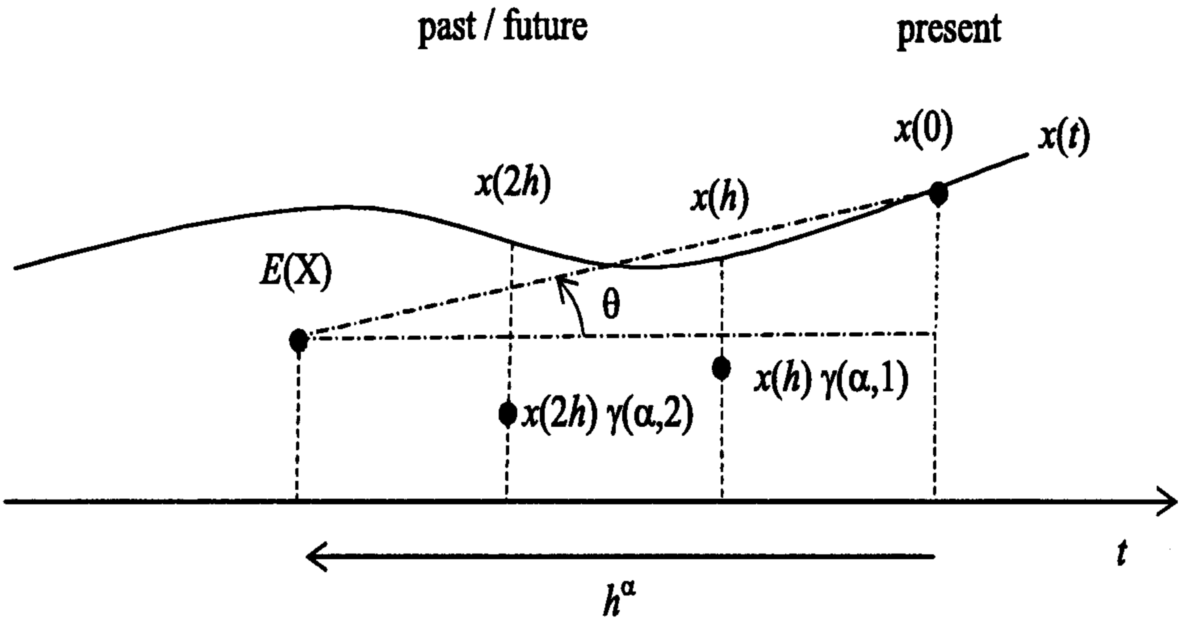

Figure 1: Geometric and probabilistic interpretation of the Grünwald-Letnikov definition of a derivative of fractional order α of the signal $x(t)$

The Grünwald-Letnikov definition (1) gets the slope θ of a triangle composed by $x(0)$ and E(X) placed at location $t=h^{\alpha}$, that is, the "present" sample of the signal x and the arithmetic average of the "past/future". As the increment $h \to 0$ the slope $\theta \to D^{\alpha}[x(t)]$. We get $D^1[x(t)]$ and $D^0[x(t)]$ for the particular cases of $\alpha = 1$ and $\alpha = 0$ corresponding to the slope of tangent line (because the "past/future" has probability one for the sample near the "present" and zero for the rest of the "past/future") and the present value of x (because all the "past/future" has probability zero) respectively. By other words, the integer cases correspond to a deterministic perspective that is just a limit situation of the more general case of a fractional value of α.

On the other hand, it is also important to analyse the amplitude of the probability distribution that captures and weights the "past/future" for getting E(X). Figure 2 depicts $|\gamma(\alpha,k)|$ *versus* k for several values of α. As $k \to +\infty$ we get the asymptotic approximation $|\gamma(\alpha,k)|$ gets proportional to $k^{-\alpha-1}$ [17] showing the well known logarithmic like memory and the importance that the fractional derivative gives to the "past/future" sample values of $x(t)$ in opposition with the case of integer order.

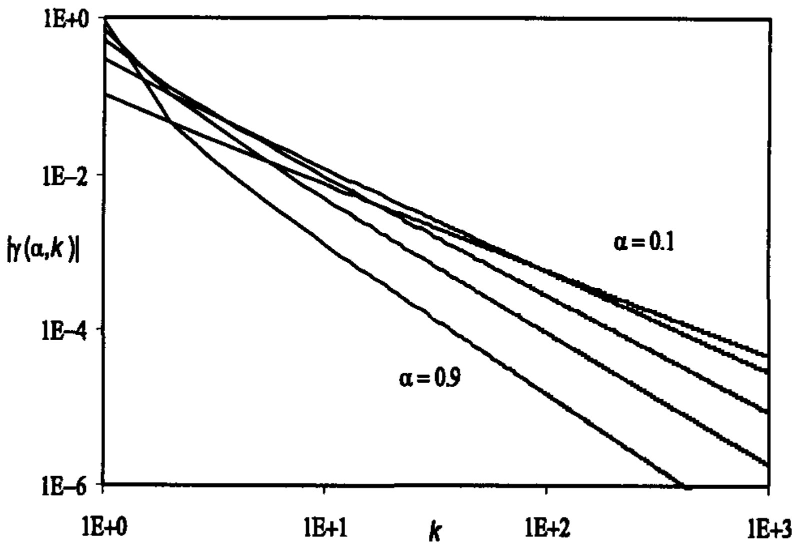

Figure 2: Amplitude $|\gamma(\alpha,k)|$ of the probability distribution *versus* k for $\alpha = \{0.1,...,0.9\}$

The same conclusions can be drawn through the arithmetic average and the variance of the probability distribution, namely $\mu_X = 2^{\alpha-1}\alpha$ and $V_X = \alpha\,(1 + \alpha - \alpha 2^{\alpha})\,2^{\alpha-2}$, as depicted in Figure 3.

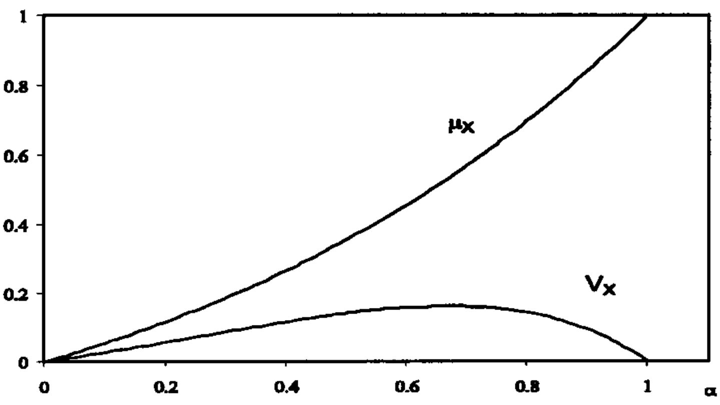

Figure 3: Arithmetic average μ_X and variance V_X of the probability distribution $|\gamma(\alpha,k)|$ *versus* α

It is interesting to note that the maximum variance occurs for $\alpha = 0.66858$ instead of $\alpha = 0.5$ because the probability distribution differs significantly from the asymptotic expansion for the first terms.

3. Conclusions

In the last years the progress in the scientific knowledge motivated the adoption of the theory of fractional calculus as a useful mathematical tool to handle applications in the areas of physics, chemistry and engineering sciences. The work carried out so far is still preliminary but reveals interesting and promising aspects for future research and developments. Nevertheless, the lack of a simple interpretation for the base concept of a derivative or an integral of non-integer order poses problems and, consequently, such limitation must be overcome.

In this line of thought, this paper presented a novel approach based on the probability theory and the Grünwald-Letnikov of a fractional order derivative. The concepts are simple and lead to a clear geometric interpretation that is compatible to established interpretations for the standard cases of integer order.

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