A POSTER ABOUT THE RECENT HISTORY
OF FRACTIONAL CALCULUS

J. A. Tenreiro Machado [1], Virginia Kiryakova [2],
Francesco Mainardi [3]

Abstract

In the last decades fractional calculus became an area of intense research and development. The accompanying poster illustrates the major contributions during the period 1966-2010.

MSC 2010: 26A33, 05C72, 33E12, 34A08, 34K37, 35R11, 60G22
Key Words and Phrases: fractional calculus, history

Preliminary note

Clearly, lists such as those assembled in the paper [1] as well as in the present short note, can never be complete, and, besides, there must be selective decisions. We do apologize for all omissions. Moreover, we have not given any judgment on the references in [1]: we limited ourselves to cite (possibly/hopefully) most of them.

The progress of fractional calculus

The fractional calculus started from some speculations of G. W. Leibniz (1695, 1697) and L. Euler (1730), and it has been developed progressively up to now. A list of mathematicians, who have provided important contributions up to the middle of the twentieth century, includes P. S. Laplace

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(1812), S. F. Lacroix (1819), J. B. J. Fourier (1822), N. H. Abel (1823-1826),
J. Liouville (1832-1873), B. Riemann (1847), H. Holmgren (1865-1867), A.
K. Grunwald (1867-1872), A. V. Letnikov (1868-1872), H. Laurent (1884),
P. A. Nekrassov (1888), A. Krug (1890), J. Hadamard (1892), O. Heaviside
(1892-1912), S. Pincherle (1902), G. H. Hardy and J. E. Littlewood (1917-
1928), H. Weyl (1917), P. Lvy (1923), A. Marchaud (1927), H.T. Davis
(1924-1936), E. L. Post (1930), A. Zygmund (1935-1945), E. R. Love (1938-
1996), A. Erdelyi (1939-1965), H. Kober (1940), D. V. Widder (1941), M.
Riesz (1949), W. Feller (1952).

Only since the Seventies the fractional calculus has been the object of
specialized conferences and treatises. For the first conference the merit
is due to B. Ross who, shortly after his Ph.D. dissertation on fractional
calculus, organized the First Conference on Fractional Calculus and its Ap-
plications at the University of New Haven in June 1974, and edited its pro-
ceedings. For the first monograph, the merit is ascribed to K. B. Oldham
and J. Spanier who, after a joint collaboration begun in 1968, published a
book devoted to fractional calculus in 1974.

In the recent years considerable interest in fractional calculus has been
stimulated by the applications it finds in different areas of applied sciences
like physics and engineering, possibly including fractal phenomena. Now
there are more books of proceedings and special issues of journals published
that refer to the applications of fractional calculus in several scientific areas
including special functions, control theory, chemical physics, stochastic pro-
cesses, anomalous diffusion, rheology. Several special issues appeared in the
last decade which contain selected and improved papers presented at con-
ferences and advanced schools, concerning various applications of fractional
calculus. Already since several years, there exist two international journals
devoted almost exclusively to the subject of fractional calculus: *Journal
of Fractional Calculus* (Editor-in-Chief: K. Nishimoto, Japan) started in
1992, and *Fractional Calculus and Applied Analysis* (Managing Editor: V.
Kiryakova, Bulgaria) started in 1998.

The authors have recently collected and listed information (see **Table
1**) about the progress in the area of fractional calculus during the period
1966-2010 in the paper [1] entitled "Recent History of Fractional Calcu-
lus" and published in *Communications in Nonlinear Science and Numeri-
cal Simulation*. The time series can be analyzed through the multidimen-
sional scaling method. Defining the time correlation comparison factor as

$$c_{ij} = \frac{\sum x_i(t)x_j(t)}{[\sum x_i(t)x_i(t) \sum x_j(t)x_j(t)]^{\frac{1}{2}}}, \quad i,j = 1, \cdots, 6,$$ the series lead to a 6×6 matrix

Table 1: Fractional Calculus during 1966-2010

Year	Special Issues	Books Edited	Books with Author	Conferences	Special Sessions	Patents
1966	-	-	1	-	-	-
1967	-	-	-	-	-	-
1968	-	-	-	-	-	-
1969	-	-	1	-	-	-
1970	-	-	-	-	-	-
1971	-	-	-	-	-	-
1972	-	-	-	-	-	-
1973	-	-	1	-	-	-
1974	-	-	1	1	-	-
1975	-	1	-	-	-	-
1976	-	-	-	-	-	-
1977	-	-	1	-	-	-
1978	-	-	1	-	-	-
1979	-	-	2	-	-	-
1980	-	-	-	-	-	-
1981	-	-	-	-	-	-
1982	-	-	2	-	-	-
1983	-	-	1	-	-	-
1984	-	-	1	1	-	-
1985	-	1	-	-	-	-
1986	-	-	1	-	-	-
1987	-	-	2	-	-	-
1988	-	-	-	-	-	-
1989	-	-	1	1	-	-
1990	-	1	-	-	-	-
1991	-	-	4	-	-	-
1992	-	-	2	-	-	-
1993	-	1	2	-	-	1
1994	-	-	2	1	-	1
1995	-	1	1	-	-	-
1996	-	-	2	1	-	-
1997	-	1	-	-	-	-
1998	-	1	1	1	-	-
1999	1	-	4	2	-	-
2000	1	1	-	-	-	-
2001	-	-	1	1	1	-
2002	3	-	5	-	2	-
2003	1	-	3	3	2	-
2004	2	-	1	1	1	-
2005	-	2	2	2	-	1
2006	1	-	2	4	2	5
2007	3	2	-	3	1	3
2008	3	1	7	3	2	1
2009	2	1	5	4	2	-
2010	2	3	6	4	5	-

and to the three dimensional map represented in **Figure 1**. We verify that, as expected, the Conferences (C), Special Sessions in conferences (SS), Special Issues in journals (SI) and Books with Author (BA) are closely related, while the Books Edited (BE) and Patents (P) have a different evolution.
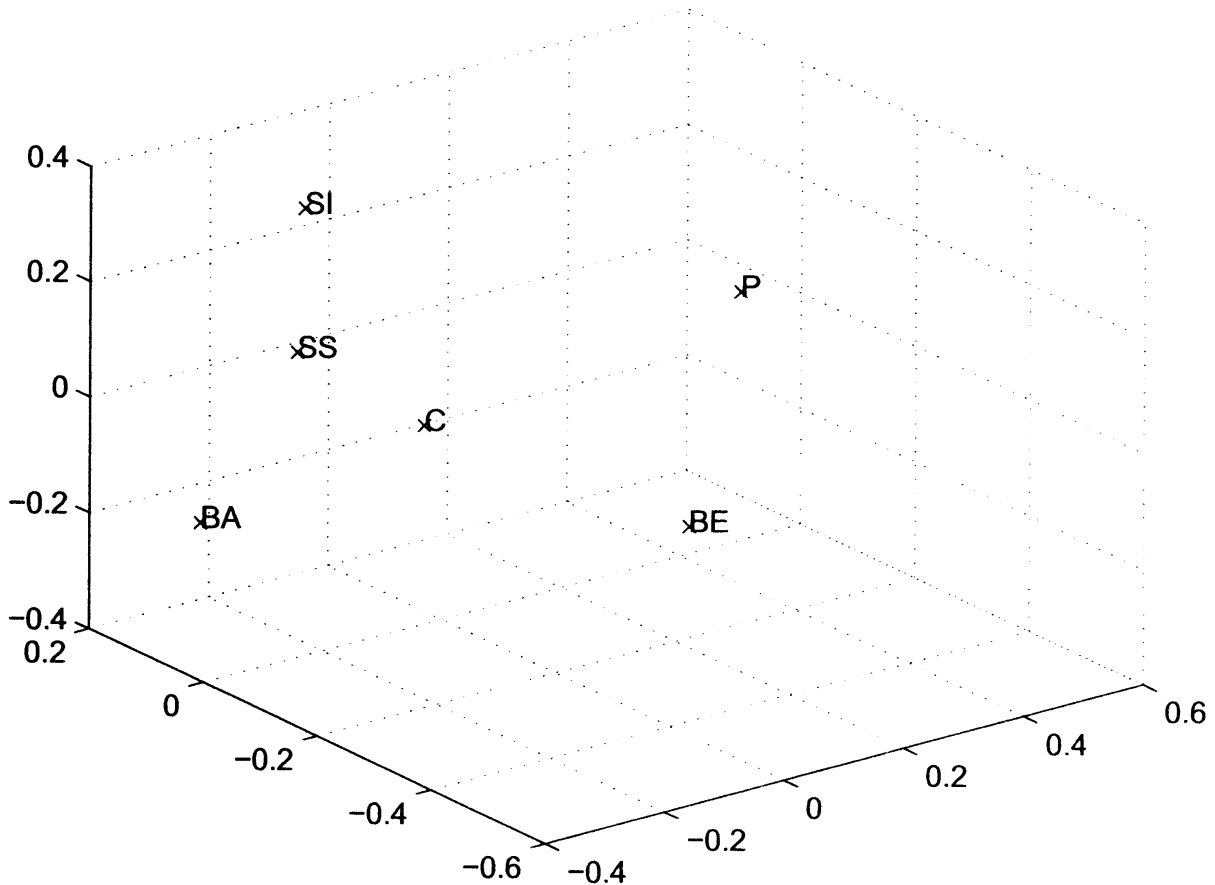

Figure 1: The multidimensional scaling map for the fractional calculus evolution during the period 1966-2010. C - Conferences, SS - Special sessions in conferences, SI - Special issues in journals, BE - Books edited, BA - Books with author, P - Patents.

This note presents a **POSTER** (reduced version is depicted at **Figure 2**, and **A3 format** in true colors is attached separately) with the time line of events in this area during 1966-2010. The authors believe that this list demonstrates the large volume of research in the area of fractional calculus that will continue to grow up in the forthcoming years, and that it will constitute an important tool in the scientific progress of mankind.
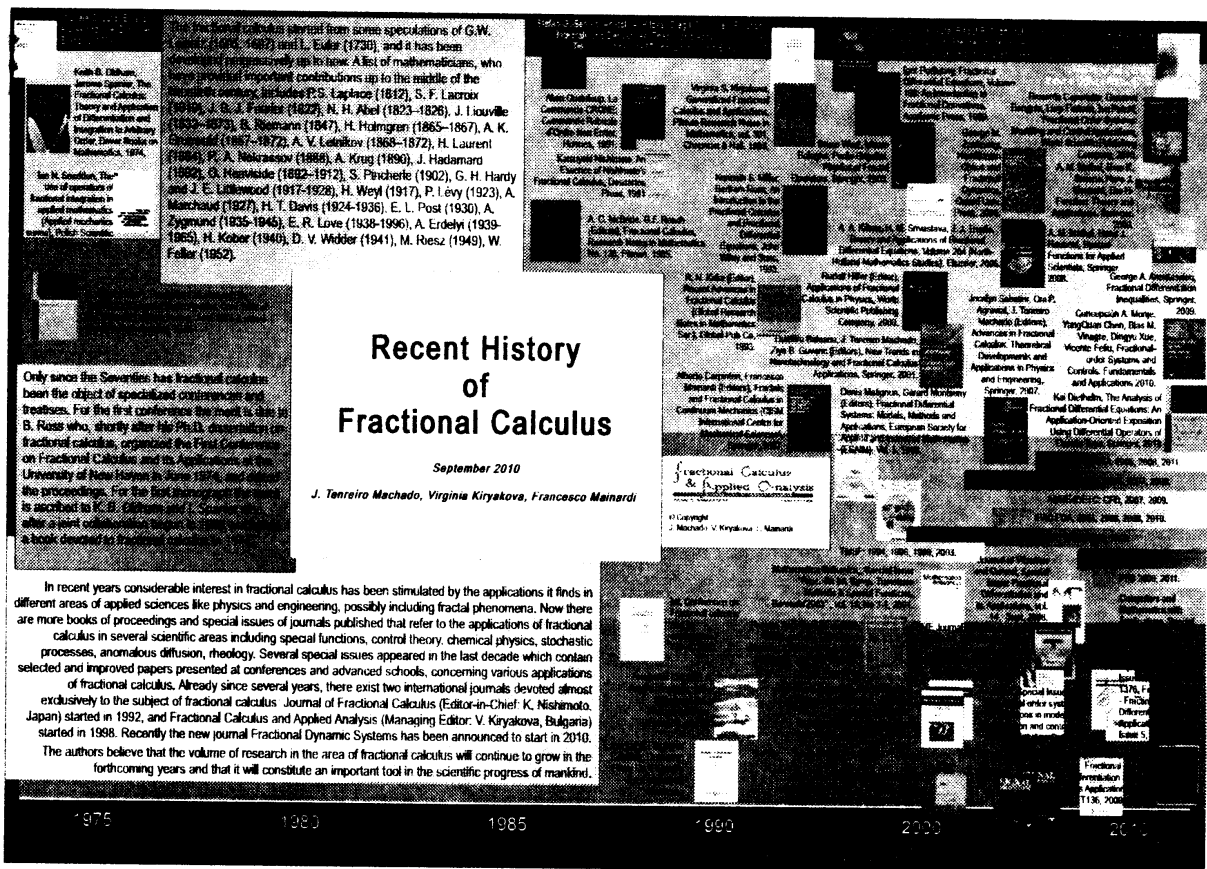
Figure 2: The time line of fractional calculus during the period 1966-2010.

Acknowledgments

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[1] J. Tenreiro Machado, Virginia Kiryakova, Francesco Mainardi, Recent History of Fractional Calculus. *Communications in Nonlinear Science and Numerical Simulation* (Elsevier) **16** (2011), 1140-1153; doi:10.1016/j.cnsns.2010.05.027

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Recent History
of
Fractional Calculus

September 2010

J. Tenreiro Machado, Virginia Kiryakova, Francesco Mainardi

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The authors believe that the volume of research in the area of fractional calculus will continue to grow in the forthcoming years and that it will constitute an important tool in the scientific progress of mankind.

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