Short communication

Implementation of fractional-order electromagnetic potential through a genetic algorithm

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A R T I C L E I N F O

A B S T R A C T

Several phenomena present in electrical systems motivated the development of comprehensive models based on the theory of fractional calculus (FC). Bearing these ideas in mind, in this work are applied the FC concepts to define, and to evaluate, the electrical potential of fractional order, based in a genetic algorithm optimization scheme. The feasibility and the convergence of the proposed method are evaluated.

1. Introduction

A closer look upon several phenomena that occur in electrical systems [1] induced a new approach using the tools of fractional calculus (FC). In fact, several researchers [2,3] verified that well-known expressions for the electrical potential are related through integer-order differential relationships and proposed their generalization based on the concept of fractional-order poles. Nevertheless, the mathematical generalization towards FC lacks a comprehensive method for its practical implementation.

This article addresses the synthesis of fractional-order multipoles. In Section 2, we recall the classical expressions for the static electric potential and we analyze them in the perspective of FC. Based on this re-evaluation we develop a GA scheme for implementing fractional-order electrical potential approximations. Finally, in Section 3, we outline the main conclusions.

2. Integer and fractional electrical potential

For homogeneous, linear and isotropic media, the electrical potential φ at a point P produced by several charge configurations, such as, a single charge, a dipole, a quadrupole, an infinite straight filament, two opposite charged filaments,

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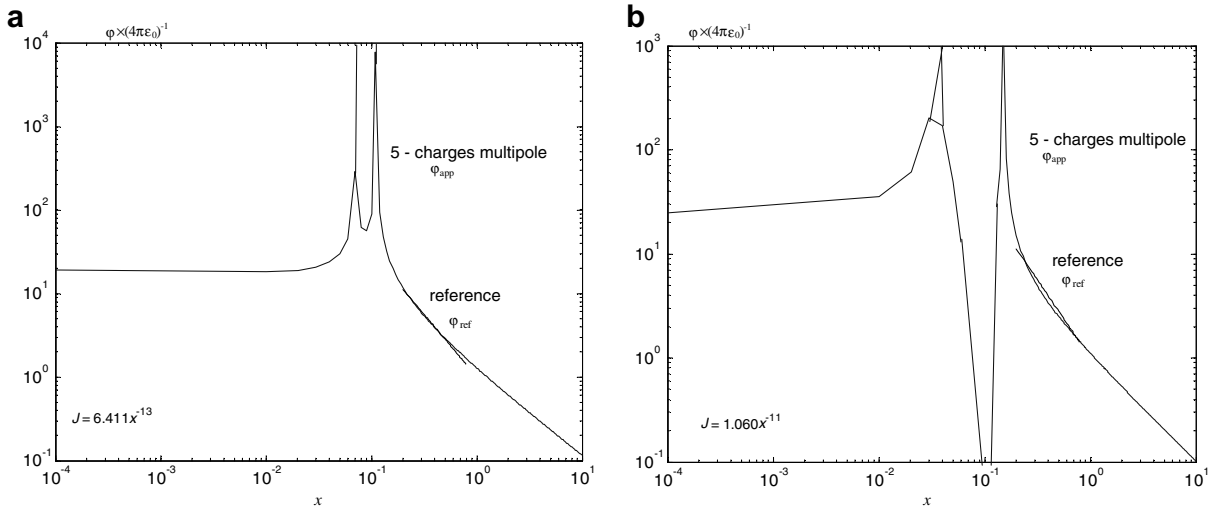
Fig. 1. Comparison of the electric potential φ_{app} and φ_{ref} versus the position x for $\varphi_{ref} = 1.0x^{-1.5}$ (V), $0.2 < x < 0.8$ [m], and a $n = 5$ charge approximation, in both cases.
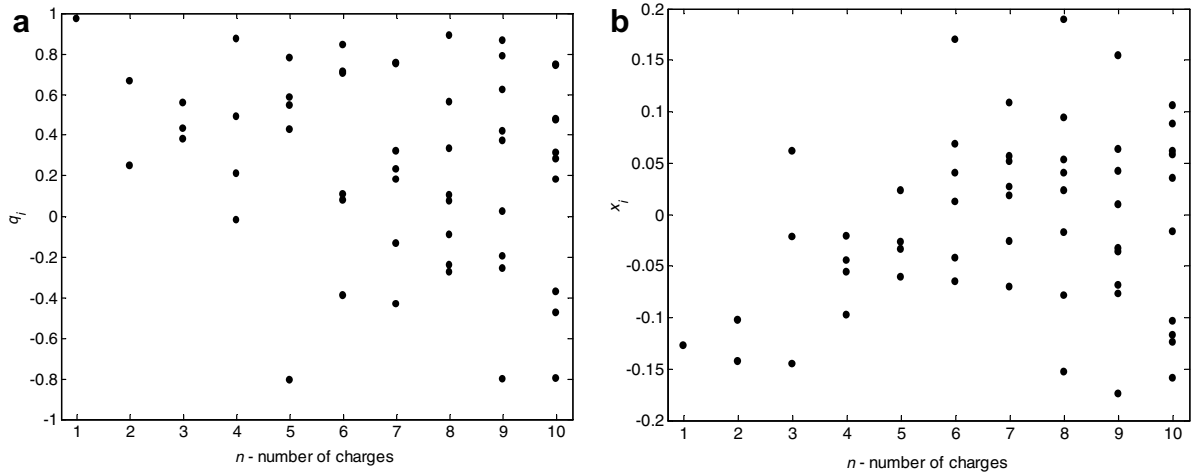

Fig. 2. Values of (a) charges q_i and the (b) corresponding positions x_i versus n, for a distribution of charges with $n = \{1, \ldots, 10\}$, $\varphi_{ref} = 1.0x^{-1.5}$ [V], $0.2 < x < 0.8$ [m], for the best case of $N_{GA} = 10$ executions.
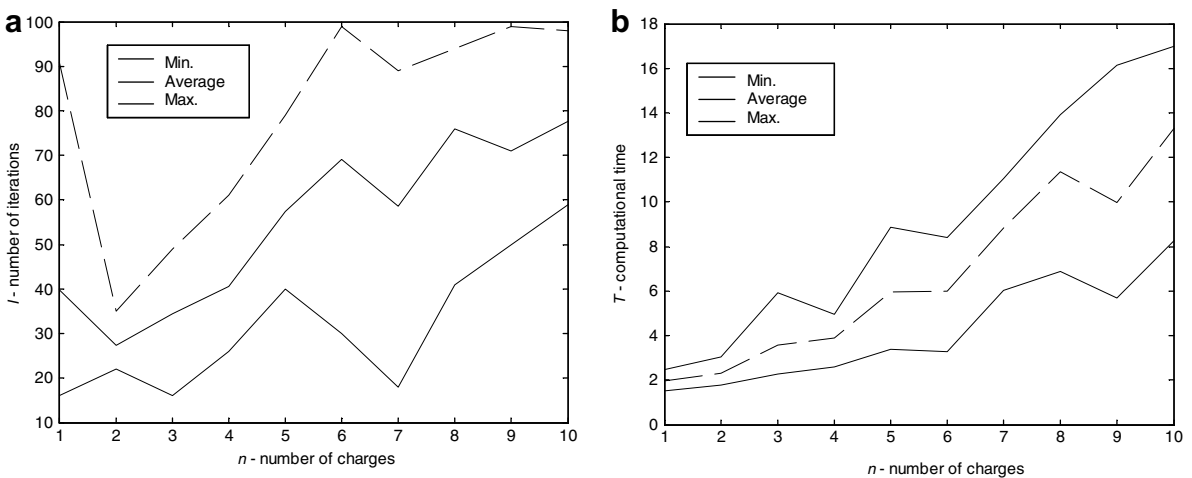

Fig. 3. Performance of the GA scheme versus the number charges $n = \{1, \ldots, 10\}$ for $\varphi_{ref} = 1.0x^{-1.5}$ [V], $0.2 < x < 0.8$ [m], (a) number of required iterations I, (b) computational time T, for $N_{GA} = 10$ executions.

and a planar surface [4], are related through the integer-order differential relationship $\varphi \sim r^{-3}, r^{-2}, r^{-1}, \ln r, r$, respectively, where r is the radial distance. This pattern of the 'integer-order multipoles' motivated several researchers [1,3] to propose its generalization in a FC perspective. Therefore, a 'fractional multipole' produces at point P a potential $\varphi \sim r^{\alpha}, \alpha \in \Re$. Nev-
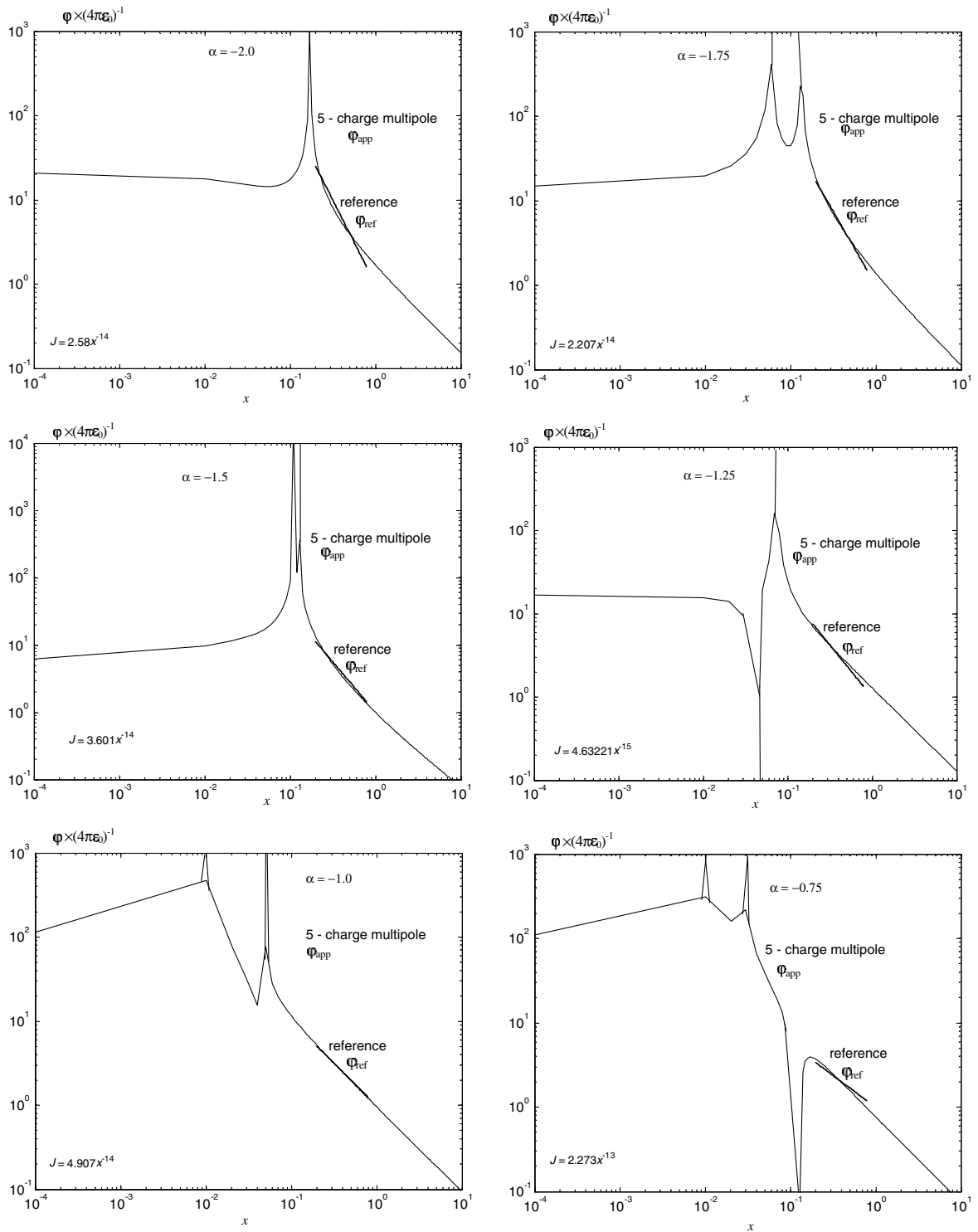
Fig. 4. Comparison of the electrical potential φ_{app} and φ_{ref} versus the position x for $\varphi_{ref} = 1.0x^{\alpha}$ [V], $\alpha = \{-2.0, -1.75, -1.5, -1.25, -1.0, -0.75, -0.5, -0.25, 0.0\}$, $0.2 < x < 0.8$ [m] and $n = 5$ charge approximation, for the best case of $N_{GA} = 10$ executions.
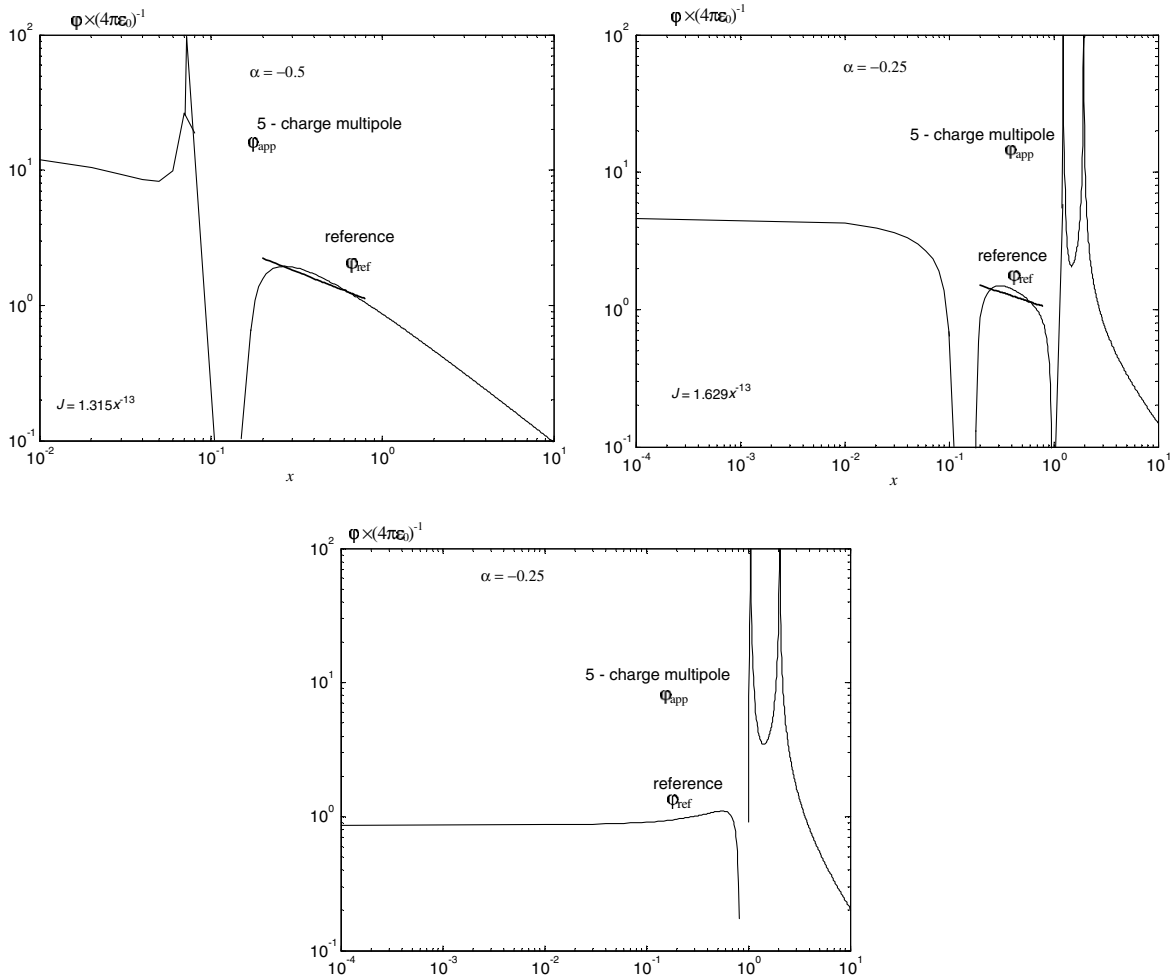
$\varphi \times (4\pi\varepsilon_0)^{-1}$

$\alpha = -0.5$

5 - charge multipole
φ_{app}

reference
φ_{ref}

$J = 1.315x^{13}$

x

$\varphi \times (4\pi\varepsilon_0)^{-1}$

$\alpha = -0.25$

5 - charge multipole
φ_{app}

reference
φ_{ref}

$J = 1.629x^{13}$

x

$\varphi \times (4\pi\varepsilon_0)^{-1}$

$\alpha = -0.25$

5 - charge multipole
φ_{app}

reference
φ_{ref}

Fig 4. (*continued*)

ertheless, besides the abstract manipulation of mathematical expressions, the truth is that no practical method, and physical interpretation, for establishing the fractional potential was developed until recently [2,3,5–7].

Inspired by the integer-order recursive approximation of fractional-order transfer functions [8,9], in this article we adopt a genetic algorithm (GA) [10–12] for implementing a fractional-order potential. It was realized that the electrical integer-order potential has a *global* nature, while the fractional-order potential has a *local* nature; therefore, it is possible to capture the fractional potential only in a restricted region of the space. This observation leads to an implementation approach, conceptually similar to the one described in [5,8,9,13], namely, to an approximation scheme based on the recursive placement of charges in appropriated locations, so that the final composition resembles the desired fractional potential.

In this line of thought, we develop a GA that determines n charges q_i and their positions x_i leading to an approximate potential φ_{app} given by

$$\varphi_{app} = \sum_{i=1}^{n} \frac{q_i}{4\pi\varepsilon_0 |x - x_i|}. \tag{1}$$

The resulting potential mimics the desired reference $\varphi_{ref} = ax^{\alpha}$, where $a \in \Re$ is a constant of proportionality and $\alpha \in \Re$ is the slope, in a given interval of approximation $x_{mim} < x < x_{max}$.

It is important to refer that a reliable execution and analysis of a GA usually requires a large number of simulations, to provide that stochastic effects are properly considered. Therefore, in this study the experiments consist in executing the GA a given number $N_{GA} \in \aleph$ times (each with a different randomly generated initial population), in order to generate a combination of positions and charges that lead to an electrical potential that approximates adequately the desire reference potential. In the first case of study, the values of GA parameters are: population number $P = 40$, crossover C (%) = 85.0%, mutation M (%) = 1.0% and an elitist strategy ES (%) = 10.0%. The chromosome has $2n$ genes: the first n genes correspond to the charges and the last n genes indicate their positions. The gene codifications adopts a Gray Code with a string length of $l = 16$ bits.

The fitness function corresponds to the minimization of the index:

$$J = \sum_{k=1}^{m} \left(\ln \left| \frac{\varphi_{\text{app}}}{\varphi_{\text{ref}}} \right| \right)^2, \quad \min_i(J), \quad i = 0, 1, \ldots, n-1, \tag{2}$$

where m is the number of sampling points, logarithmically distributed along the interval $x_{\text{mim}} < x < x_{\text{max}}$. We establish a maximum number of iterations I_{max} and a stopping scheme when $J < J_{\text{max}}$ for the best individual (i.e., the solution) of the GA population.

Fig. 1a depicts φ_{app} for a $n = 5$ charges and $\varphi_{\text{ref}} = 1.0x^{-1.5}$, $0.2 < x < 0.8$ for $I_{\text{max}} = 100$ and $J_{\text{max}} < 10^{-10}$. The GA solution consists in the set of charges $\{q_1, q_2, q_3, q_4, q_5\} = \{0.032, 0.765, 0.307, 0.282, -0.231\}$ [C] (with scale factor $\times(4\pi\varepsilon_0)^{-1}$) located at $\{x_1, x_2, x_3, x_4, x_5\} = \{-0.116, -0.110, -0.071, 0.028, 0.095\}$ [m], respectively. In this case, the GA needs $I = 39$ iterations to satisfy the adopted fitness function stopping threshold.

Different GA executions show not only a good fit between φ_{app} and φ_{ref}, but also that it is possible to find more than one 'good' solution. For example, Fig. 1b depicts another solution obtained for $I = 82$ iterations, leading to $\{q_1, q_2, q_3, q_4, q_5\} = \{0.817, 0.159, -0.451, -0.481, 0.971\}$ [C] (with scale factor $\times(4\pi\varepsilon_0)^{-1}$) located at $\{x_1, x_2, x_3, x_4, x_5\} = \{-0.151, -0.127, -0.123, -0.076, -0.035\}$ [m]. Nevertheless, for a given application, a superior precision may be required and, in that case, a larger number of charges must be used. In this line of thought, we study the performance of this method for different number of charges, namely from $n = 1$ up to $n = 10$ charges, and we compare the required number of GA iterations when the number of charges increases. In order to analyze the precision of this distribution of charges, we study the number of iterations I and the computational time T, when varying the number of charges in the set $n = \{1, \ldots, 10\}$.

Fig. 2 shows the values of the charges q_i and the corresponding positions x_i, $n = \{1, \ldots, 10\}$, for the best case within a sample of $N_{\text{GA}} = 10$ experiments. We verify that the value of the charge and their location does not 'converge' to a clear pattern.
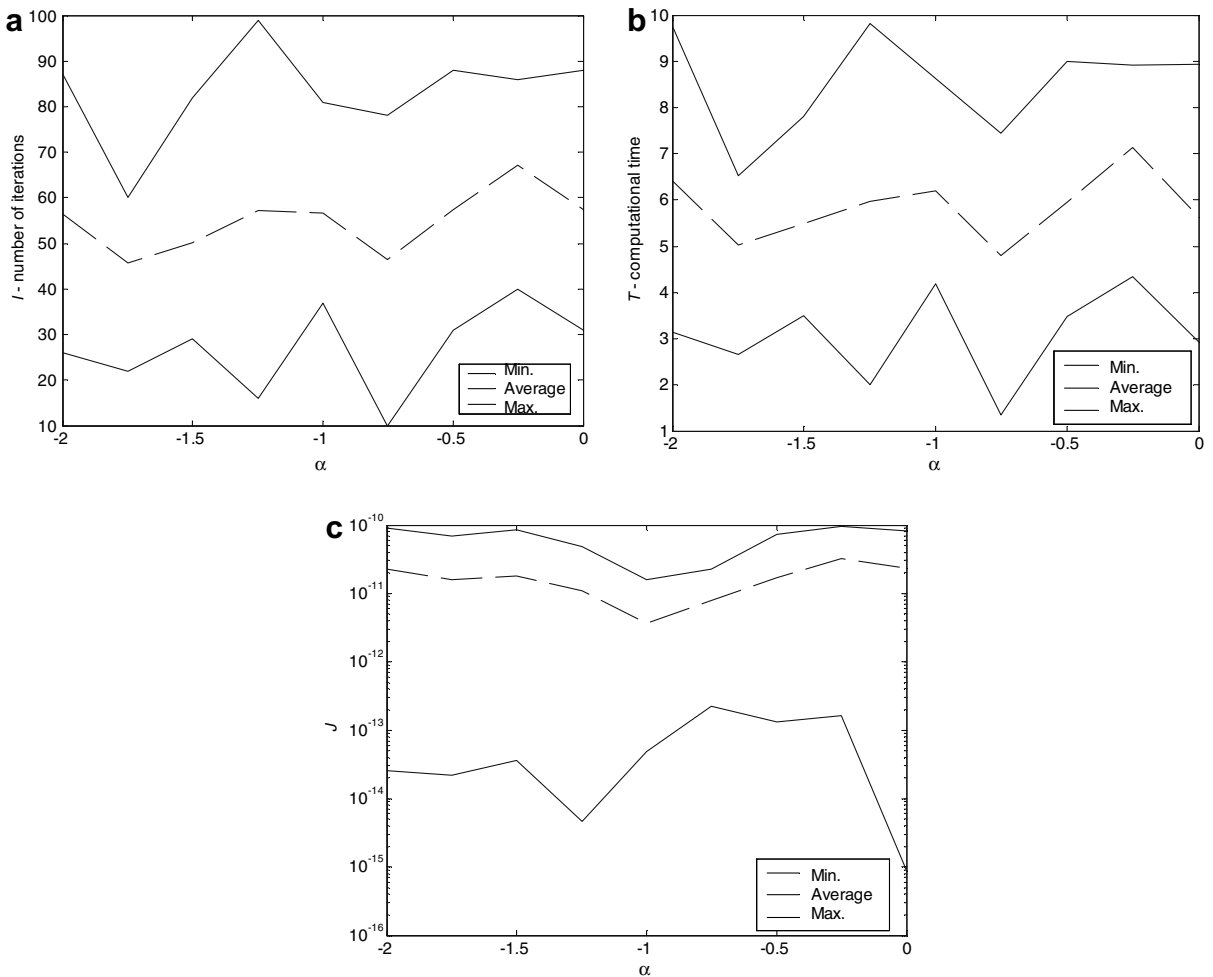

Fig. 5. Performance of the GA scheme versus α for $0.2 < x < 0.8$ [m], (a) number of required iterations I, (b) computational time T, (c) error J, for $n = 5$ charges, and $N_{\text{GA}} = 10$ executions.

Fig. 3a depicts the minimum, average and maximum of the number of required GA iterations I versus n. This chart reveals clearly that the required number of iterations increases with n.

We can also evaluate the GA computational time T for different number of charges. Therefore, we test the GA scheme for identical parameters and fitness function. Fig. 3b illustrates the corresponding minimum, average and maximum of T versus n. We verify that we get a smaller approximation error J, but we need larger computational times T, the larger the value of n.

With the proposed method it is also possible to have a reference potential with other values α for the slope. Fig. 4 shows a five charge approximation for $0.2 < x < 0.8$ and $\varphi_{\text{ref}} = 1.0x^{\alpha}$, when $\alpha = \{-2.0, -1.75, -1.5, -1.25, -1.0, -0.75, -0.5, -0.25, 0.0\}$. Again, due to the stochastic nature of the GA are considered $N_{\text{GA}} = 10$ executions and in Fig. 4 is depicted the best solution.

The number of iterations I, the computational time T and the error J, reveal a smooth evolution with α. Fig. 5 illustrates the minimum, average and maximum of I, T and J versus α, for the $N_{\text{GA}} = 10$ cases.

In conclusion, the fit between φ_{app} and φ_{ref} is adequate, but there is no obvious pattern for the charge distribution as n increases. This lack of 'order' is due to the large number of possible solutions. Therefore, the GA has a high freedom to search through within the solution space, choosing values that are almost not correlated. However, we believe that further study, imposing more strict restrictions, may lead to the emergence of a comprehensive pattern.

3. Conclusions

This paper addressed the problem of implementing a fractional-order electric potential through a genetic algorithm. The results reveal the necessity of a larger number of iterations when the number of charges increases. The GA reveals a good compromise between the accuracy and computational time. The GA approach constitutes a step towards the development of a simple design technique and, consequently, several of its aspects must be further evaluated.

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