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AN EVOLUTIONARY APPROACH FOR THE SYNTHESIS OF FRACTIONAL POTENTIALS

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Dedicated to Professor Paul L. Butzer on the occasion of his 80th birthday

Abstract

The problem of point multipoles with electrical charges and the corresponding potential is an important subject in the field of electromagnetism. In this work, it is applied the concept of fractional calculus to define, and to evaluate, the electrical potential of fractional order based in a genetic algorithm (GA) optimization scheme.

Mathematics Subject Classification: 26A33 (main), 78A25, 93A30
Key Words and Phrases: electrical potential, fractional order systems, electromagnetism, multipoles, genetic algorithms, optimization

1. Introduction

A new look of several phenomena present in electrical systems, such as motors, transformers and lines [1], induced an approach based in the fractional calculus (FC) viewpoint. Some authors [2],[3] verified that well-known expressions for the electrical potential are related through integer-order integral and derivatives, and have proposed its generalization based on the concept of fractional-order poles. Nevertheless, the mathematical generalization towards FC lacks a comprehensive method for its practical implementation.

Bearing these ideas in mind, in this article we address the synthesis of fractional-order multipoles. In Section 2 we recall the classical expressions for the static electric potential and we analyze them in the perspective of FC. Based on this re-evaluation in Section 3 we develop a GA scheme for implementing fractional-order electrical potential approximations. Finally, in Section 4 we outline the main conclusions.

2. Classical expressions for the static electrical potential

For a homogeneous, linear and isotropic media, the electric potential φ at a point P produced by a single charge (1a), a dipole (1b), a quadrupole (1c), an infinite straight filament carrying a charge λ per unit length (2a), two opposite charged filaments (2b), and a planar surface with charge density σ (3), are given by [4], [5]:

$$\varphi = \frac{q}{4\pi\varepsilon_0}\frac{1}{r} + C\,, \tag{1a}$$

$$\varphi = \frac{ql\cos\theta}{4\pi\varepsilon_0}\frac{1}{r^2} + C,\ r >> l\,, \tag{1b}$$

$$\varphi = \frac{ql^2\left(3\cos^2\theta - 1\right)}{4\pi\varepsilon_0}\frac{1}{r^3} + C,\ r >> l\,, \tag{1c}$$

$$\varphi = -\frac{\lambda}{2\pi\varepsilon_0}\ln r + C\,, \tag{2a}$$

$$\varphi = \frac{\lambda l\cos\theta}{2\pi\varepsilon_0}\frac{1}{r} + C,\ r >> l\,, \tag{2b}$$

$$\varphi = -\frac{\sigma}{2\varepsilon_0}r + C\,, \tag{3}$$

where $C \in \Re$, ε_0 represents the permittivity, q the electric charge, r the radial distance and θ the corresponding angle with the axis.

Analyzing expressions (1)-(3) we verify the relationship $\varphi \sim r^{-3}, r^{-2}, r^{-1}$, $\ln r, r$, corresponds to the application of integer-order derivatives and integrals.

3. Implementation of the fractional potential

The integer-order differential nature of the potential expressions (1)-(3) motivated several authors (see [3]) to propose its generalization in a FC perspective. Therefore, a fractional multipole produces at point P a potential $\varphi \sim r^{\alpha},\ \alpha \in \Re$. Nevertheless, besides the abstract manipulation of mathematical expressions, the truth is that there is no practical method, and physical interpretation, for establishing the fractional potential [2], [3], [6], [7], [8].

Inspired by the integer-order recursive approximation of fractional-order transfer functions [12], [13], in this section we adopt a genetic algorithm (GA) [9], [10], [11] for implementing a fractional order potential.

We start by re-evaluating the potential produced at point $P \equiv (x,y)$ by a straight filament with finite length l and charge q:

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{q}{l}\ln\left\{\frac{y+\ \frac{1}{2}l+\ \sqrt{x^2+\ \left(y+\ \frac{1}{2}l\right)^2}}{y-\frac{1}{2}l+\ \sqrt{x^2+\ \left(y-\frac{1}{2}l\right)^2}}\right\}+C,\ \ C\in\Re\,. \qquad (4)$$

It is well-known that for $x \to \infty$ we have $\varphi \to \frac{q}{4\pi\varepsilon_0}\frac{1}{x}+C$ and, with $y = 0$, for $x \to 0$ we have $\varphi \to \frac{1}{2\pi\varepsilon_0}\frac{q}{l}\ln\left(\frac{1}{x}\right)+C$. Obviously these limit cases correspond to (1a) and (2a) respectively, that is, to a single charge and to an infinite long filament.

Figure 1 depicts the potential (4) versus x (with $l = 1\ m$ and $y = 0$) and, for comparison, the limit cases (1a) and (2a) (for $C = 0$).

In this chart we observe that expression (4) changes smoothly between the two limit cases. Therefore, we can have an intermediate fractional-order relationship as long as we restrict to a limited working range. For example, for $0.1 < x < 0.4$ and $0.3 < x < 1.0$ we get the approximations $\varphi \approx 1.292x^{-0.569}$ and $\varphi \approx 0.9825x^{-0.821}$, with squared coefficient of determination $R^2 = 0.993$ and $R^2 = 0.998$, respectively.

This means that standard integer-order potential relationships have a *global* nature while fractional-order potentials have a *local* nature possible to capture only in a restricted region. This conclusion leads to an implementation approach conceptually similar to the one described in [12], [13] that is, to an approximation scheme based on a recursive placement of integer-order functions. Nevertheless, in the present case we do not have the analytical formalism of Bode diagrams and, therefore, we need to adopt a numerical approach.

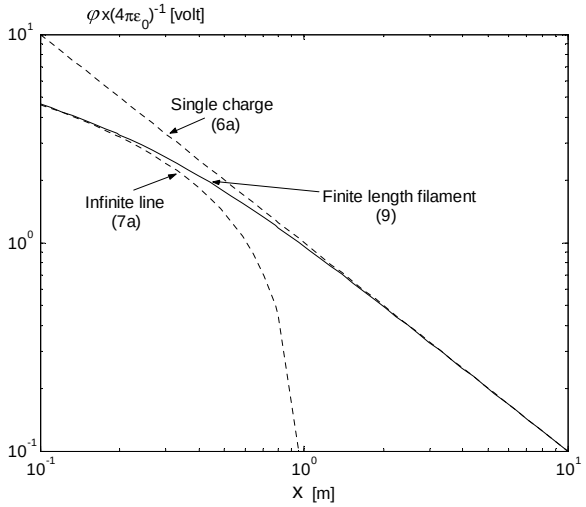

Figure 1: Comparison of the electric potential φ *versus* x for a filament (4) with charge $q = 1$, length $l = 1\ m$, at $y = 0$, a single charge (1a) and an infinite line (2a)

In this line of thought, we develop a one-dimensional GA that determines the values of charges and the corresponding positions. Two different situations are analyzed. In Subsection 3.1 the positions of charges are located symmetrically, and in Subsection 3.2 the charges are located asymmetrically.

3.1. Symmetrical distribution of charges

In this subsection, the GA places recursively n charges q_i ($i = 0, \ldots, (n-1)/2$, $n-$odd; $i = 1, \ldots, n/2$, $n-$even) at the symmetrical positions $\pm\, x_i$ (with exception of $x_0 = 0$ that corresponds to the center of the n-array of charges, n – odd, where there is a single charge q_0).

Our goal is to compare the approximate potential φ_{app}, resulting from a number of charges and the corresponding locations, with the desired reference potential $\varphi_{ref} = kx^{\alpha}$:

$$\varphi_{app} = \begin{cases} \frac{q_0}{|x|} + \sum_{i=1}^{\frac{n-1}{2}} \frac{q_i}{4\pi\varepsilon_0} \left(\frac{1}{|x-x_i|} + \frac{1}{|x+x_i|} \right) & n \text{ odd} \\ \sum_{i=1}^{\frac{n}{2}} \frac{q_i}{4\pi\varepsilon_0} \left(\frac{1}{|x-x_i|} + \frac{1}{|x+x_i|} \right) & n \text{ even} \end{cases} . \qquad (5)$$

It is important to refer that a reliable execution and analysis of a GA usually requires a large number of simulations to provide that stochastic effects have been properly considered [14], [15], [16], [17]. Therefore, in this study the experiments consist on executing the GA several times, in order to generate a combination of positions and charges that lead to an electrical potential with fractional slope similar to the desire reference potential. In the first case of study, the values of GA parameters are: population number P = 40, crossover C(%) = 85.0%, mutation M(%) = 1.0% and an elitist strategy ES(%) = 10.0%. We establish a maximum number of iterations $I_{Max} = 100$ and a stoping scheme when $J < 10^{-5}$ for the best individual (*i.e.*, solution) of the GA population. The optimization fitness function corresponds to the minimization of the index:

$$J = \sum_{k=1}^{m} \left(\ln \left| \frac{\varphi_{app}}{\varphi_{ref}} \right| \right)^2, \quad \min_i (J), \quad i = 0, 1, ..., n-1, \tag{6}$$

where m is the number of sampling points along the x-axis.

For example, **Figure 2** shows a $n = 5$ charge approximation for $\varphi_{ref} = 1.0\, x^{-1.5}$, $0.2 < x < 0.8$, for two different GA solutions:

- case A: $q_{0A} = -0.489$ [C], $q_{1A} = 0.920$ [C] and $q_{2A} = -0.077$ [C] (with scale factor $\times(4\pi\varepsilon_0)^{-1}$), located at $x_{0A} = 0.0$ [m], $x_{1A} = \pm 0.147$ [m] and $x_{2A} = \pm 0.185$ [m], respectively;

- case B: $q_{0B} = +0.280$ [C], $q_{1B} = +0.161$ [C] and $q_{2B} = +0.361$ [C] (with scale factor $\times(4\pi\varepsilon_0)^{-1}$), located at $x_{0B} = 0.0$ [m], $x_{1B} = \pm 0.103$ [m] and $x_{2B} = \pm 0.159$ [m], respectively.

In the case A the GA needs $I_A = 32$ iterations to satisfy the fitness function, and in the case B the GA needs $I_B = 36$ iterations.

The results show a good fit between φ_{ref} and φ_{app} and it is clear that it is possible to find more than one 'good' solution. Nevertheless, for a given application, a superior precision may be required and, in that case, a larger number of charges must be used. In this line of thought, we study the performance of this method for different number of charges, namely from $n = 1$ up to $n = 10$ charges, and we compare the necessary number of GA iterations when the number of charges increases.

Figure 3 shows the positions of the charges, for $n = 1$ up to $n = 10$, and the corresponding values (represented by circles of proportional size).

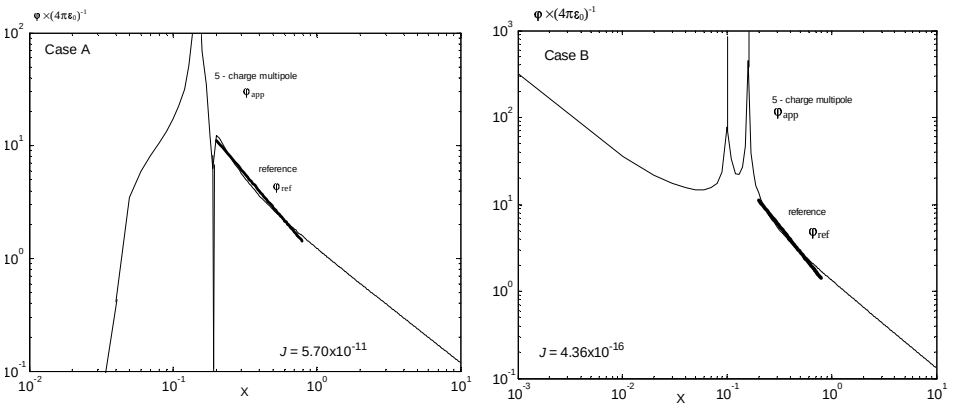

Figure 2: Comparison of the electrical potential φ_{app} and φ_{ref} *versus* the position x for $\varphi_{ref} = 1.0\ x^{-1.5}$ [volt], $0.2 < x < 0.8$ [m], and a $n = 5$ symmetrical charge approximation.

We verify that the charge *versus* the location pattern is not clear and its comparison with a fractal or a recursive layout is not straightforward.

Figure 4a) depicts the minimum, average and maximum of the number of required GA iterations I *versus* n. This chart reveals clearly that the number of iterations increases with n. In order to evaluate the GA computational time T (in seconds) for different number of charges, we test the GA scheme for identical parameters, {P = 40, C(%) = 85.0%, M(%) = 1.0%, ES(%) = 10.0%, $I = 100$} and the fitness function J given by equation (6). **Figure 4b)** illustrates the corresponding minimum, average and maximum of T *versus* n, and confirms the previous conclusions.

3.2. Asymmetrically distribution of charges

In this subsection, the GA places, asymmetrically, the n charges and determines the corresponding values q_i. The electrical potential is now described by the equation:

$$\varphi_{app} = \sum_{i=1}^{n} \frac{q_i}{4\pi\varepsilon_0 \left| x - x_i \right|} \,. \tag{7}$$

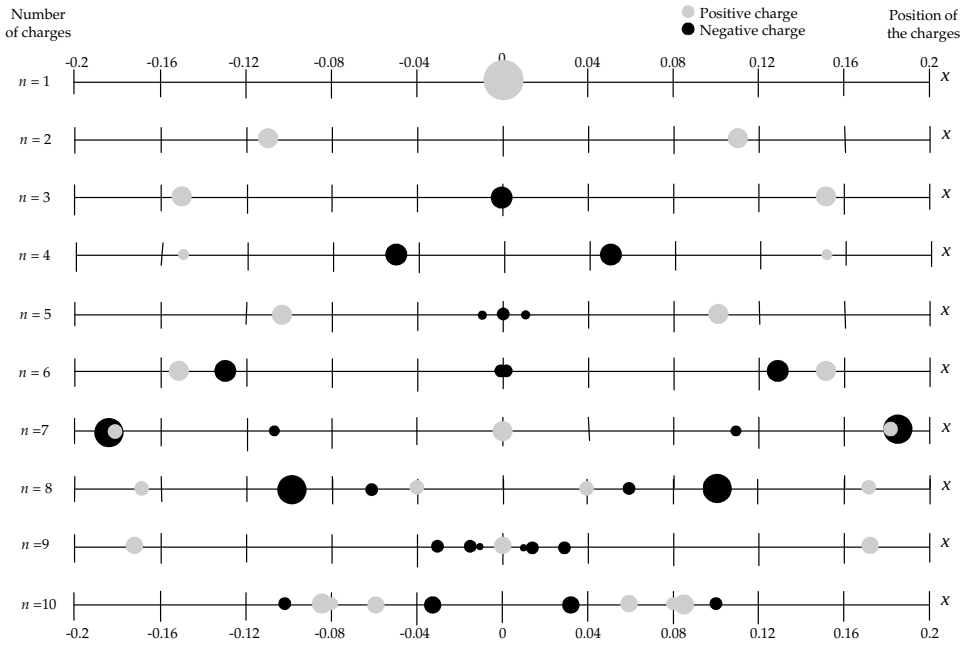

Figure 3: Positions of charges x_i and the corresponding values q_i for a symmetrical distribution of charges with $n = \{1, \ldots, 10\}$, $\varphi_{ref} = 1.0\ x^{-1.5}$ [volt], $0.2 < x < 0.8$ [m].

The optimization fitness function J is identical to the previous case and is given by equation (6). The values of GA parameters are also identical $\{\mathrm{P} = 40, \mathrm{C}(\%) = 85.0\%, \mathrm{M}(\%) = 1.0\%, \mathrm{ES}(\%) = 10.0\%, I_{Max} = 100\}$. The study is similar to the one developed in the previous subsection, namely the determination of the error J, the number of required iterations I and the computational time T needed for calculating the electrical potential φ *versus* the number of charges n.

Figure 5 shows a pre-defined number of $n = 5$ charge approximation and $\varphi_{ref} = 1.0\ x^{-1.5}$, $0.2 < x < 0.8$, leading to $q_1 = 0.880$ [C], $q_2 = 0.283$ [C], $q_3 = 0.154$ [C], $q_4 = -0.823$ [C] and $q_5 = 0.333$ [C] (with scale factor $\times(4\pi\varepsilon_0)^{-1}$), located at $x_1 = -0.130$ [m], $x_2 = -0.106$ [m], $x_3 = 0.01$ [m], $x_4 = 0.025$ [m] and $x_5 = 0.055$ [m], respectively. In this case, the GA needs $I = 51$ iterations to satisfy the fitness function.

The results show a good fit between φ_{ref} and φ_{app} and, again, it is possible to find more than one 'good' solution. In order to analyze the

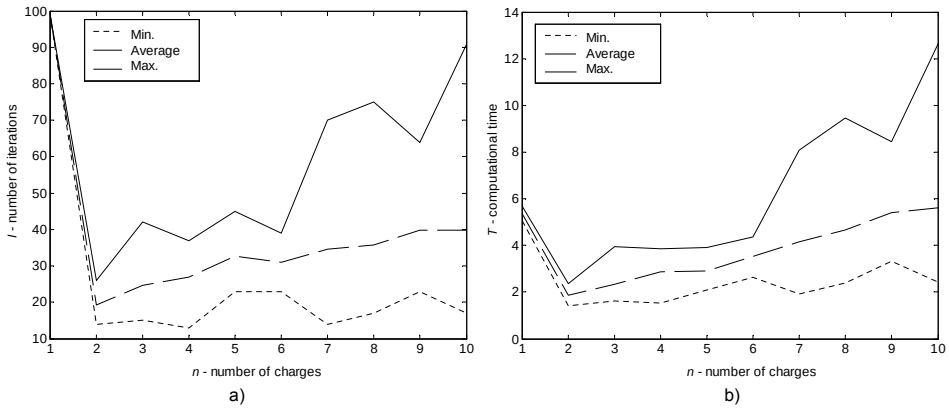

Figure 4: Performance of the GA scheme *versus* the number charges n, $n = \{1, \ldots, 10\}$ for $\varphi_{ref} = 1.0\ x^{-1.5}$ [volt], $0.2 < x < 0.8$ [m], for a symmetrical distribution, a) number of iterations I, b) Computational time T.

precision of this distribution of charges, we study the require number of iterations I and the computational time T when the number of charges varies from $n = 1$ up to $n = 10$.

Once more, the charges *versus* location, **Figure 6**, seems not to converge to any clear pattern.

Figures 7a) and **7b)** depict the minimum, average and maximum of the number of iterations I and the computational time T *versus* n, respectively.

When we compare these results with those presented in the previous subsection, we verify that we get a smaller approximation error J but a larger computational time T, for all values of n. Furthermore, the number of iterations I increases significantly when the GA distributes asymmetrically the charges due to the larger optimization burden.

4. Conclusions

This paper addressed the problem of implementing a fractional-order electric potential. The results reveal the necessity of a larger number of iterations when the number of charges increases, and consequently a larger value of T. We conclude also that when the GA distributes the electrical

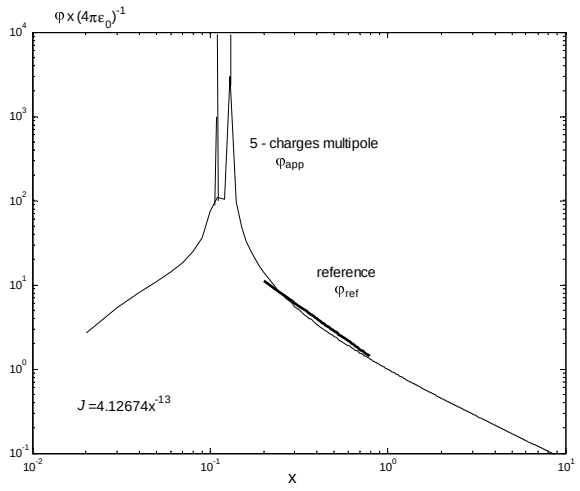

Figure 5: Comparison of the electric potential φ_{app} and φ_{ref} *versus* the position x for $\varphi_{ref} = 1.0\ x^{-1.5}$ [volt], $0.2 < x < 0.8$ [m], and a $n = 5$ asymmetrical charge approximation.

charges asymmetrically the approximation error decreases but, due to the larger number of variables, the number of required iterations increases. The GA reveals a good compromise between the accuracy and computational time. The GA approach constitutes a step towards the development of a simple design technique and, consequently, several of its aspects must be further evaluated.

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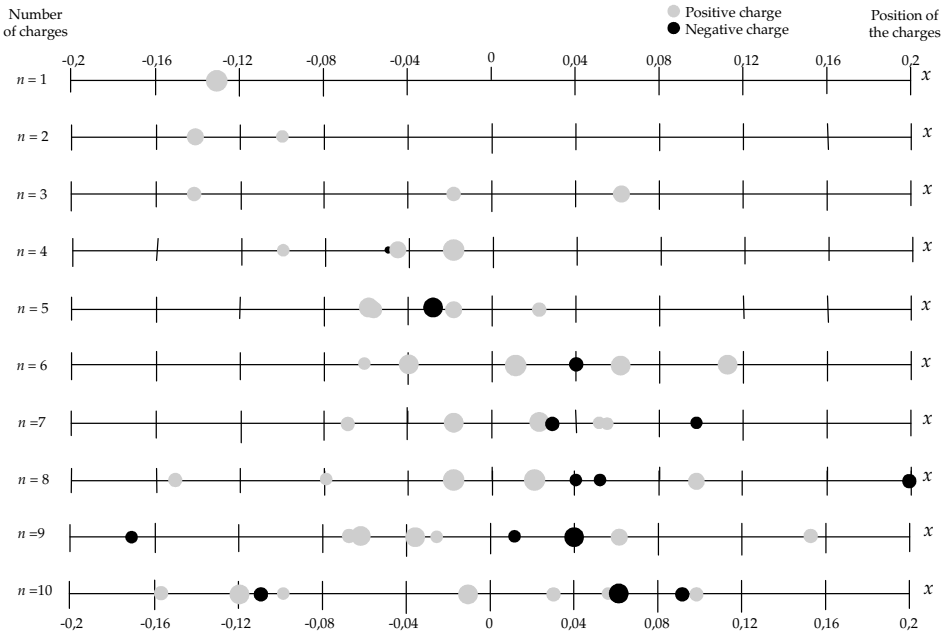

Figure 6: Positions of charges x_i and the corresponding values q_i for a asymmetrical distribution of charges with $n = \{1, \ldots, 10\}$, $\varphi_{ref} = 1.0\ x^{-1.5}$ [volt], $0.2 < x < 0.8$ [m].

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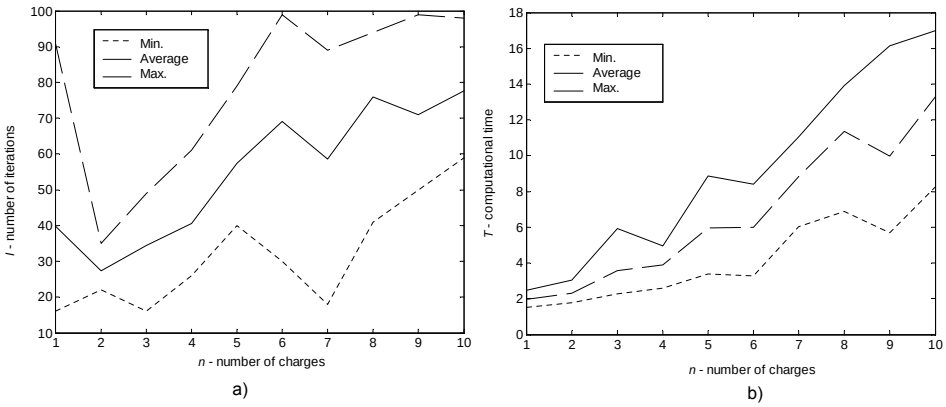

Figure 7: Performance of the GA scheme *versus* the number charges n, $n = \{1, \ldots, 10\}$ for $\varphi_{ref} = 1.0\ x^{-1.5}$ [volt], $0.2 < x < 0.8$ [m], a) number of required iterations I, b) computational time T, for a asymmetrical distribution.

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