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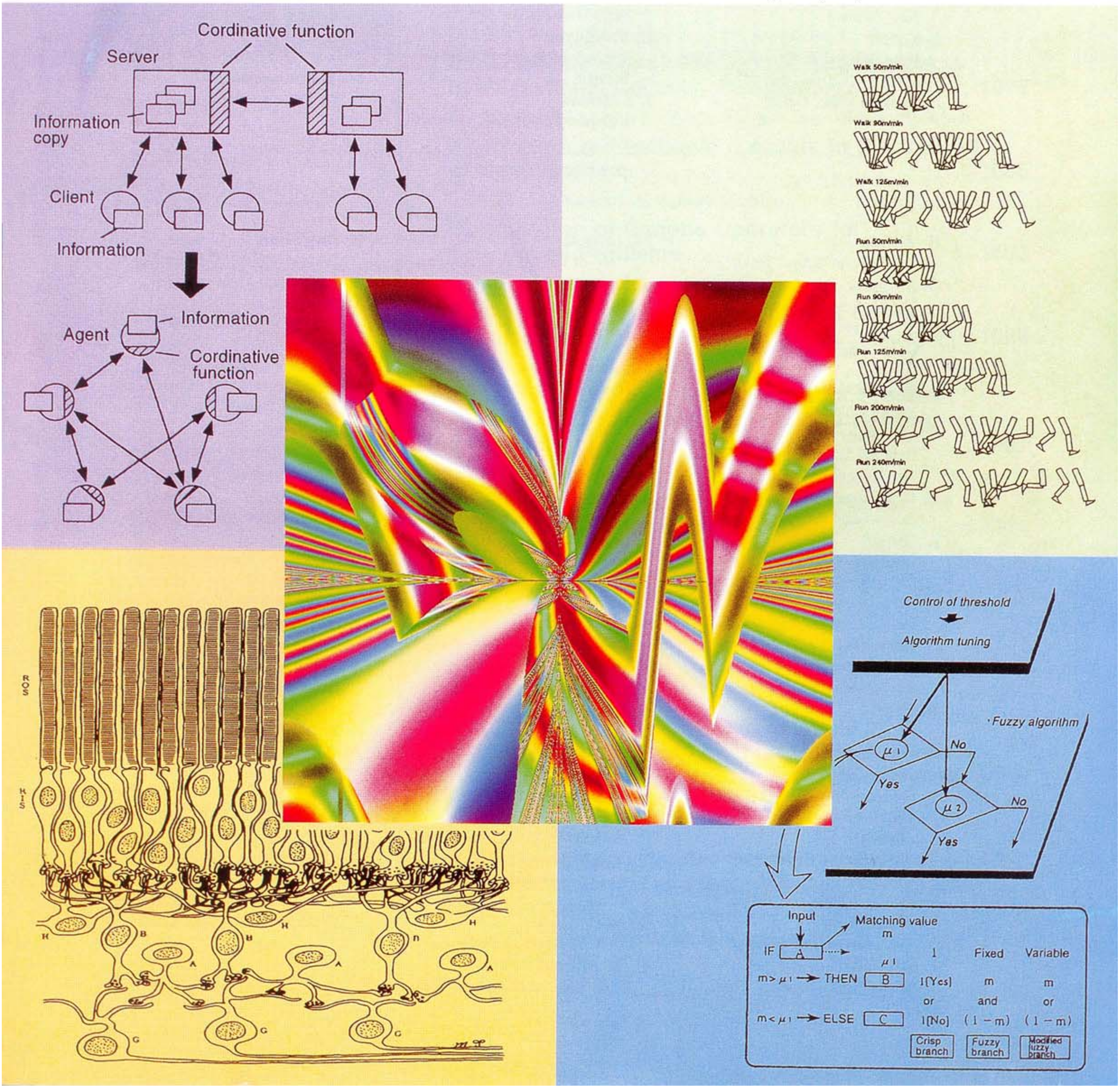

Paper:

Application of Fractional Calculus in the Control of Heat Systems

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The PID controller is by far the most dominating form of feedback in use in the process industries, due to its functional simplicity and performance. In this work, we apply a generalization of the PID, namely the fractional controller PID$^{\beta}$, to the heat diffusion system. For the PID$^{\beta}$ tuning are used four performance indices, to find the optimum controller settings by taking advantage of the fractional order β. The effect of actuator saturation and the required control energy are also analyzed.

Keywords: fractional calculus, control, diffusion systems, ISE, ITSE, IAE, ITAE

1. Introduction

Fractional calculus (FC) is a generalization of integration and differentiation to a non-integer order $\alpha \in C$, being the fundamental operator ${}_aD_t^{\alpha}$, where a and t are the limits of the operation [1, 2].

In the last years, FC has been used increasingly to model the constitutive behavior of materials and physical systems exhibiting hereditary and memory properties. This is the main advantage of fractional derivatives in comparison with classical integer models, where these effects are simply neglected. It is well-known that the fractional-order operator $s^{0.5}$ appears in several types of problems. The transmission lines, heat flow or the diffusion of neutrons in a nuclear reactor are examples where the half-operator is the fundamental element. On the other hand, diffusion is one of the three fundamental partial differential equations of mathematical physics [3].

In this paper we investigate the heat diffusion system in the perspective of applying the FC theory. A fractional-order PID algorithm is presented and compared with the classical scheme. The fractional PI$^{\alpha}$D$^{\beta}$ controller involves an integrator of order $\alpha \in \Re^{+}$ and a differentiator of order $\beta \in \Re^{+}$.

Bearing these ideas in mind, the paper is organized as follows. Section 2 gives the fundamentals of fractional-order control systems. Section 3 introduces the heat diffusion system and describes its simulation. Section 4 points out several control strategies for the heat system and discusses the results. Finally, section 5 draws the main conclusions and addresses perspectives towards future developments.

2. Fractional-Order Control Systems

Fractional controllers are characterized by differential equations that have, in the dynamical system and/or in the control algorithm, an integral and/or a derivative of fractional-order. Due to the fact that these operators are defined by irrational continuous transfer functions, in the Laplace domain, or infinite dimensional discrete transfer functions, in the Z domain, we often encounter evaluation problems in the simulations. Therefore, when analyzing fractional systems, we usually adopt continuous or discrete integer-order approximations of fractional-order operators [4–6].

The mathematical definition of a fractional derivative and integral has been the subject of several different approaches [1, 2]. One commonly used definition is given by the Riemann-Liouville expression ($\alpha > 0$):

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \qquad n-1 < \alpha < n \quad . . . . (1) $$

where $f(t)$ is the applied function and $\Gamma(x)$ is the Gamma function of x. Another widely used definition is given by the Grünwald-Letnikov approach ($\alpha \in \Re$):

$$ {}_aD_t^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\left[\frac{t-a}{h}\right]} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} f(t-kh) \quad (2) $$

where h is the time increment and $[x]$ means the integer part of x.

The "memory" effect of these operators is demonstrated by Eqs. (1) and (2), where the convolution integral in Eq. (1) and the infinite series in Eq. (2), reveal the unlimited memory of these operators, ideal for modelling hereditary and memory properties in physical systems and materials.

An alternative definition to Eqs. (1) and (2), which reveals useful for the analysis of fractional-order control systems, is given by the Laplace transform method. Considering vanishing initial conditions, the fractional *differintegration* is defined in the Laplace domain, $F(s) =$

$L\{f(t)\}$, as:

$$L\{{}_aD_t^\alpha f(t)\} = s^\alpha F(s), \quad \alpha \in \Re. \quad . . . . . (3)$$

An important aspect of fractional-order algorithms can be illustrated through the elemental control system, with open-loop transfer function $G(s) = Ks^{-\alpha}$ $(1 < \alpha < 2)$ in the forward path. The open-loop Bode diagrams of amplitude and phase have correspondingly a slope of -20α dB/dec and a constant phase of $-\alpha\pi/2$ rad over the entire frequency domain. Therefore, the closed-loop system has a constant phase margin of $\mathrm{PM} = \pi(1-\alpha/2)$ rad, that is independent of the system gain K, and the closed-loop system is robust against gain variations exhibiting step responses with an iso-damping property [7, 8].

In this paper we adopt discrete integer-order approximations to the fundamental element s^α $(\alpha \in \Re)$ of a fractional-order control (FOC) strategy. The usual approach for obtaining discrete equivalents of continuous operators of type s^α adopts the Euler, Tustin and Al-Alaoui generating functions.

It is well known that rational-type approximations frequently converge faster than polynomial-type approximations and have a wider domain of convergence in the complex domain. Thus, by using the Euler operator $w(z^{-1}) = (1-z^{-1})/T$, and performing a power series expansion of $[w(z^{-1})]^\alpha = [(1-z^{-1})/T]^\alpha$ gives the discretization formula corresponding to the Grünwald-Letnikov definition (2):

$$D^\alpha(z^{-1}) = \left(\frac{1-z^{-1}}{T}\right)^\alpha = \sum_{k=0}^{\infty} h^\alpha(k) z^{-k} \quad . . . (4)$$

$$h^\alpha(k) = \left(\frac{1}{T}\right)^\alpha \binom{k-\alpha-1}{k} \quad . . . . (5)$$

A rational-type approximation can be obtained by applying the Padé approximation method to the impulse response sequence (5) $h^\alpha(k)$, yielding the discrete transfer function:

$$H(z^{-1}) = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}} = \sum_{k=0}^{\infty} h(k) z^{-k} \quad (6)$$

where $m \le n$ and the coefficients a_k and b_k are determined by fitting the first $m+n+1$ values of $h^\alpha(k)$ into the impulse response $h(k)$ of the desired approximation $H(z^{-1})$. Thus, we obtain an approximation that has a perfect match to the desired impulse response $h^\alpha(k)$ for the first $m+n+1$ values of k. Note that the above Padé approximation is obtained by considering the Euler operator but the determination process will be exactly the same for other types of discretization schemes.

3. Heat Diffusion

The heat diffusion is governed by a linear unidimensional partial differential equation (PDE) of the form:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad . . . . (7)$$

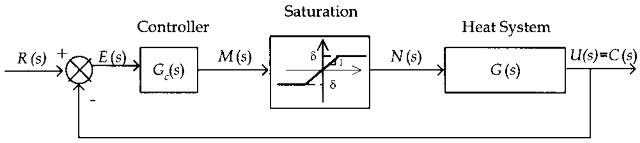

Fig. 1. Closed-loop system with PID controller G_c.

where k is the diffusivity, t is the time, u is the temperature and x is the space coordinate. The system (7) involves the solution of a PDE of parabolic type for which the standard theory guarantees the existence of a unique solution [9].

For the case of a planar perfectly isolated surface we usually apply a constant temperature U_0 at $x = 0$ and analyzes the heat diffusion along the horizontal coordinate x. Under these conditions, the heat diffusion phenomenon is described by a non-integer order model:

$$U(x,s) = \frac{U_0}{s} G(s), \quad G(s) = e^{-x\sqrt{\frac{s}{k}}} \quad . . . . (8)$$

where x is the space coordinate, U_0 is the boundary condition and $G(s)$ is the system transfer function.

In our study, the simulation of the heat diffusion is performed by adopting the Crank-Nicholson implicit numerical integration based on the discrete approximation to differentiation as [10]:

$$\begin{aligned} -ru[j+1,i+1] + (2+r)u[j+1,i] - ru[j+1,i-1] \\ = ru[j,i+1] + (2-r)u[j,i] + u[j,i-1] \quad . . . (9) \end{aligned}$$

where $r = k\Delta t(\Delta x^2)^{-1}$, $\{\Delta x, \Delta t\}$ and $\{i, j\}$ are the increments and the integration indices for space and time, respectively [11].

4. Control Strategies

The generalized PID controller $G_c(s)$ has a transfer function of the form:

$$G_c(s) = \frac{M(s)}{E(s)} = K_p\left[1 + \frac{1}{T_i s^\alpha} + T_d s^\beta\right] \quad . . . (10)$$

where α and β are the orders of the fractional integrator and differentiator, respectively. The constants K_p, T_i and T_d are correspondingly the proportional gain, the integral time constant and the derivative time constant.

Clearly, taking $(\alpha,\beta) = \{(1,1),(1,0),(0,1),(0,0)\}$ we get the classical {PID, PI, PD, P} controllers, respectively. The $\mathrm{PI}^\alpha\mathrm{D}^\beta$ controller is more flexible and gives the possibility of adjusting more carefully the closed-loop system characteristics.

In the sequel, we analyze the system of **Fig. 1** by adopting a fractional PID^β tuned by the minimization of an integral performance index.

In a previous work was demonstrated that the PID^β controller applied to an heat system reveals better results than the classical PID controller tuned through the Ziegler-Nichols open loop (ZNOL) heuristic [12, 13]. In fact, the ZNOL does not produce satisfactory results giving a significant overshoot, a large settling time and a time

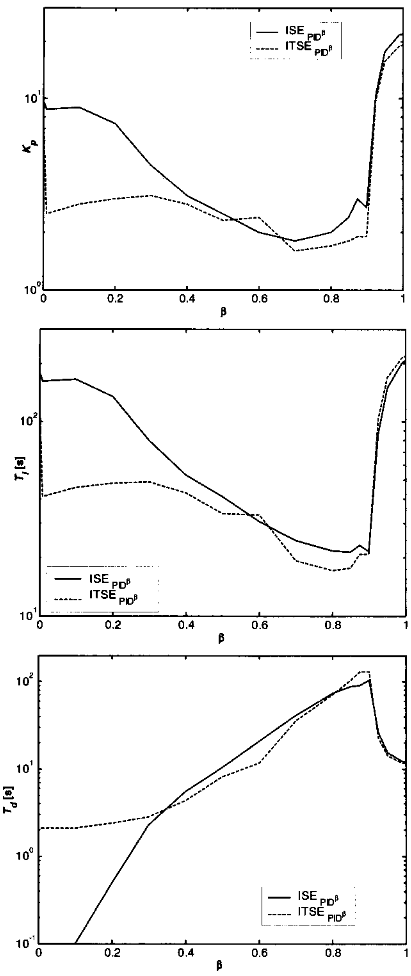

Fig. 2. The PID$^\beta$ parameters (K_p, T_i, T_d) versus β for the ISE and ITSE criteria.

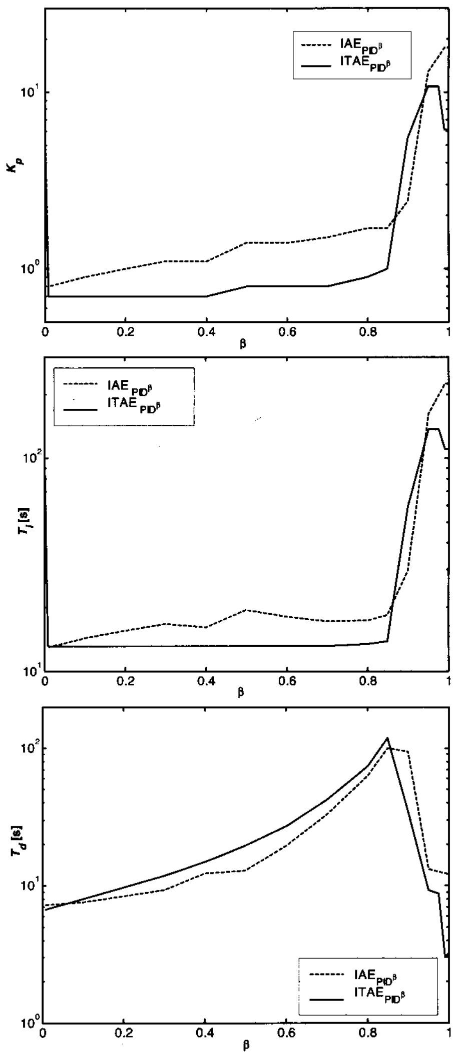

Fig. 3. The PID$^\beta$ parameters (K_p, T_i, T_d) versus β for the IAE and ITAE criteria.

delay. The fractional dynamics of the system points out other strategies, namely the adoption of fractional control algorithms.

In this sub-section we analyze the closed-loop system under the action of the fractional PID$^\beta$ controller given by the transfer function (10) with $\alpha = 1$ and $0 \le \beta \le 1$. The fractional derivative term $T_d s^\beta$ in expression (10) is implemented through a 4$^{\text{th}}$-order Padé discrete rational transfer function of type (6) and it is used a sampling period of $T = 0.1$ s.

The PID$^\beta$ controller is tuned by minimizing a performance index. We analyze and compare four indices that measure the response error, namely the integral square error (ISE), the integral time square error (ITSE), the integral absolute error (IAE) and the integral time absolute error (ITAE) criteria defined as:

$$\text{ISE} = \int_0^\infty [r(t) - c(t)]^2 dt \quad \ldots\ldots\ldots \quad (11)$$

$$\text{ITSE} = \int_0^\infty t\,[r(t) - c(t)]^2 dt \quad \ldots\ldots\ldots \quad (12)$$

$$\text{IAE} = \int_0^\infty |r(t) - c(t)|\,dt \quad \ldots\ldots\ldots \quad (13)$$

$$\text{ITAE} = \int_0^\infty t\,|r(t) - c(t)|\,dt. \quad \ldots\ldots\ldots \quad (14)$$

Another important performance index consists on the energy E_m at the PID$^\beta$ controller output $m(t)$ given by the expression:

$$E_m = \int_0^{T_e} m^2(t)dt \quad \ldots\ldots\ldots\ldots \quad (15)$$

where T_e is the time window needed to stabilize the systems output $c(t)$.

A step reference input $R(s) = 1/s$ is applied at $x = 0.0$ m and the output $u(t) = c(t)$ is analyzed for $x = 3.0$ m, without actuator saturation. The heat system is simulated for 3000 seconds. **Figs. 2 and 3** illustrates the variation of the fractional PID parameters (K_p, T_i, T_d) as function of

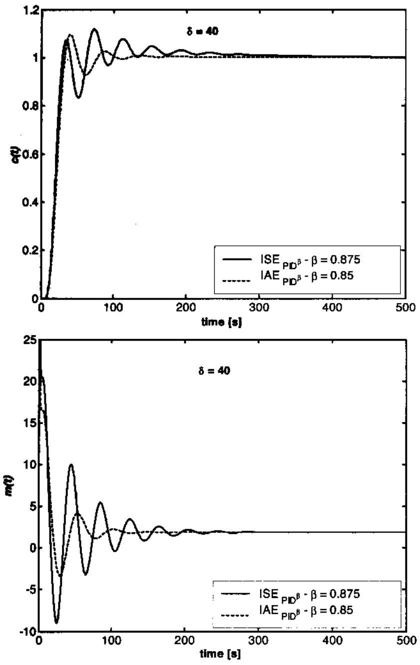

Fig. 4. Step responses of the closed-loop system and the controller output for the ISE and the IAE indices, with a PID^β controller, $\delta = 40$ and $x = 3.0$ m.

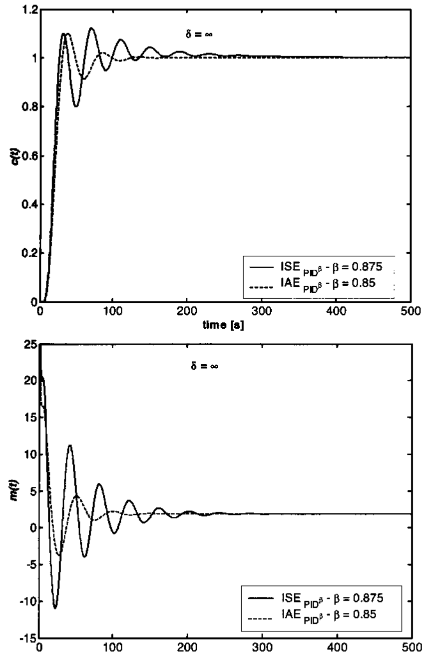

Fig. 5. Step responses of the closed-loop system and the controller output for the ISE and the IAE indices, with a PID^β controller, $\delta = \infty$ and $x = 3.0$ m.

the order's derivative β, for the criteria (11)-(14).

In **Fig. 2** the curves reveal that for $\beta < 0.4$ the parameters (K_p, T_i, T_d) are slightly different, for the two ISE and ITSE criteria, while for $\beta \geq 0.4$ they lead to almost similar values. This fact indicates a large influence of a weak order derivative on system's dynamics. However, for the criteria IAE and the ITAE the curves reveal almost similar values.

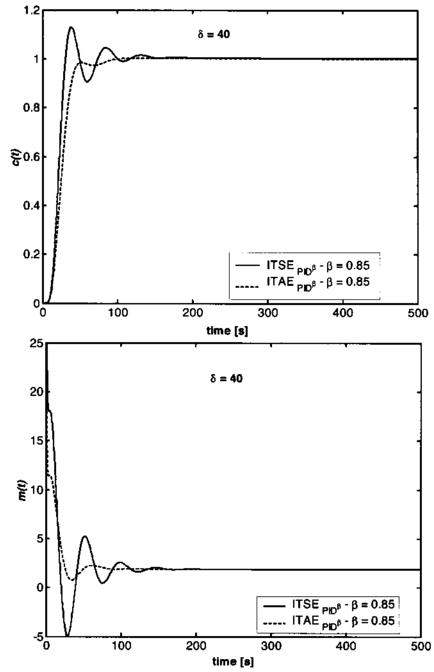

Fig. 6. Step responses of the closed-loop system and the controller output for the ITSE and the ITAE indices, with a PID^β controller, $\delta = 40$ and $x = 3.0$ m.

To further illustrate the performance of the fractional-order controllers a saturation nonlinearity is inserted in series with the controller $G_c(s)$ of **Fig. 1**, yielding:

$$n(m) = \begin{cases} m, & |m| < \delta \\ \delta\, sign(m), & |m| \geq \delta \end{cases} \quad \cdots\cdots (16)$$

where *sign(m)* is the signal function.

The controller performance is evaluated for $\delta = \{40, \ldots, 100\}$ and $\delta = \infty$ which corresponds to a system without saturation. For the PID^β we use the same parameters obtained previously, without considering the saturation.

Figures 4 and **5** depict the step responses $c(t)$ of the closed-loop system and the corresponding controller output $m(t)$, for the PID^β tuned in the ISE and IAE perspectives, with $\delta = 40$ and $\delta = \infty$, respectively. The controller tuning yields the parameters ISE: $\{K_p, T_i, T_d, \beta\} \equiv \{3, 23, 90.6, 0.875\}$ and IAE: $\{K_p, T_i, T_d, \beta\} \equiv \{1.7, 18.3, 99.9, 0.85\}$. **Figs. 6** and **7** show the corresponding step responses for the PID^β tuned according with the ITSE and ITAE for $\delta = 40$ and $\delta = \infty$, respectively. In these cases, the controller parameters yield ITSE: $\{K_p, T_i, T_d, \beta\} \equiv \{1.8, 17.6, 103.6, 0.85\}$ and ITAE: $\{K_p, T_i, T_d, \beta\} \equiv \{1.0, 13.8, 119.0, 0.85\}$.

The step response resulting from the minimization of ISE for $\delta = \infty$ reveals small overshoot and rise time, but a poor settling time, when compared with the IAE case. Both indices lead to a zero steady-state error. Nevertheless, the controller output for the ISE reveals larger variations than those occurring for the IAE. For a satura-

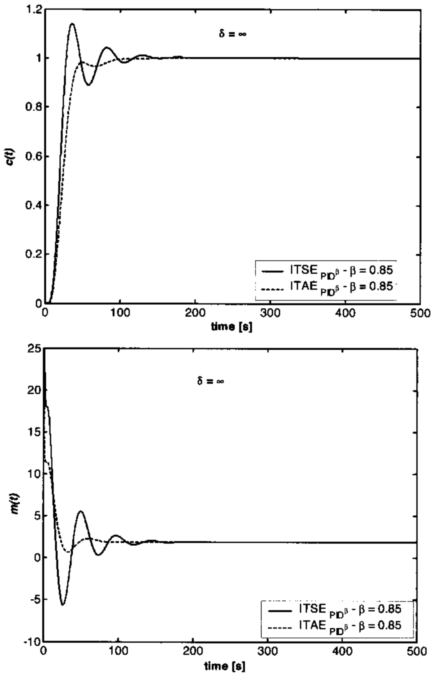

Fig. 7. Step responses of the closed-loop system and the controller output for the ITSE and the ITAE indices, with a PID$^{\beta}$controller, $\delta = \infty$ and $x = 3.0$ m.

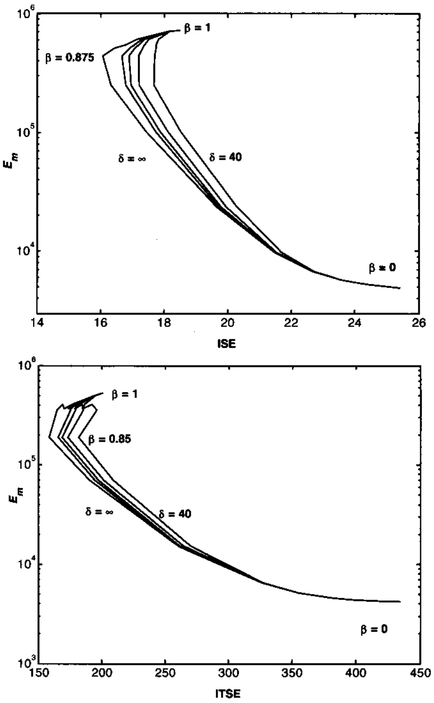

Fig. 8. Control action energy E_m for the ISE and ITSE indices *versus* $0 \leq \beta \leq 1$, when $\delta = \{40, \ldots, 100\}$ and $\delta = \infty$.

tion level of $\delta = 40$ we verify similar conclusions; however, the overshoot, the settling time and the amplitude of the controller output decreases smoothly. When we analyze the ITSE and the ITAE indices, both for $\delta = 40$ and $\delta = \infty$, we can draw similar conclusions to those obtained for the ISE and the IAE, but we verify an improvement of all the parameters of the transient and steady state responses, namely a diminishing of the settling time for both cases.

In conclusion, the IAE reveals a better transient response than the one obtained through the ISE. The step response and the controller output are also improved when the saturation level δ is diminished. Moreover, for the IAE the step response has almost no overshoot.

Figures 8 and **9** depicts the energy of the control action E_m as function of the ISE and the ITSE, or the IAE and the ITAE indices, when $0 \leq \beta \leq 1$, for $\delta = \{40, \ldots, 100\}$ and $\delta = \infty$, respectively. As can be seen, the energy E_m for ISE increases rapidly for $0 \leq \beta \leq 0.875$, while for $\beta > 0.875$ the energy increases smoothly. In the ITSE case the same conclusions can be outlined for $\beta = 0.85$.

Similar conclusions can be draw for the IAE and the ITAE indices, both for $\beta = 0.85$. For the IAE and the ITAE criteria the values of these indices and the variation of E_m versus δ is more pronounced.

In conclusion, for $0.85 \leq \beta \leq 0.875$ we get the best controller tuning, superior to the performance revealed by the classical integer-order scheme studied in [8, 9].

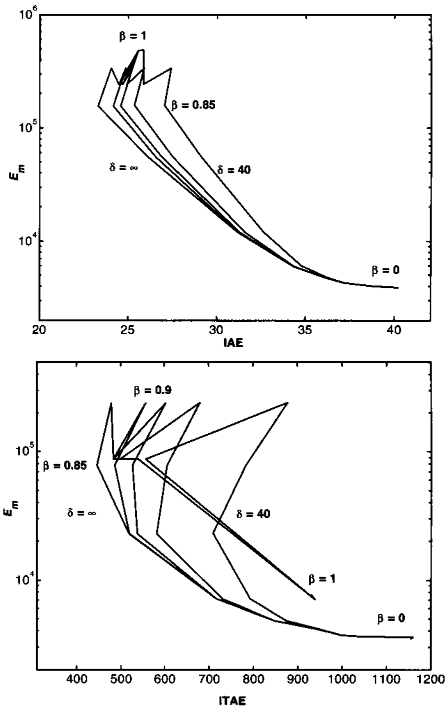

Fig. 9. Control action energy E_m for the IAE and ITAE indices *versus* $0 \leq \beta \leq 1$, when $\delta = \{40, \ldots, 100\}$ and $\delta = \infty$.

5. Conclusions

This paper presented the fundamental aspects of the FC theory and demonstrated that FC is a modelling paradigm allowing a deeper understanding of physical phenomena.

In this perspective, we studied the heat diffusion system and its control using classical and fractional PID schemes. The results show the superior performance of the FC based algorithm. The PID$^\beta$ tuning according with the ISE, ITSE, IAE, ITAE performance indices lead to good responses in the time domain. The energy index of the control action, points out a slight superior performance of the ISE and the ITSE indices. With these results, it can be establish a tradeoff between a fast transient response, a small overshoot, or a low energy of the control action.

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