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Fractional order electromagnetics

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Abstract

The Maxwell equations constitute a formalism for the development of models describing electromagnetic phenomena. The four Maxwell laws have been adopted successfully in many applications and involve only the integer order differential calculus. Recently, a closer look for the cases of transmission lines, electrical motors and transformers, that reveal the so-called skin effect, motivated a new perspective towards the replacement of classical models by fractional-order mathematical descriptions. Bearing these facts in mind this paper addresses the concept of static fractional electric potential. The fractional potential was suggested some years ago. However, the idea was not fully explored and practical methods of implementation were not proposed. In this line of thought, this paper develops a new approximation algorithm for establishing the fractional order electrical potential and analyzes its characteristics.

Keywords: Fractional calculus; Electric potential

1. Introduction

The Maxwell equations play a fundamental role in the well established formulation of the electromagnetic theory [1]. These equations lead to the derivation of precise mathematical models useful in many applications in physics and engineering. The Maxwell equations involve only the integer-order calculus and, therefore, it is natural that the resulting classical models adopted in electrical engineering reflect this perspective.

Recently, a closer look of some phenomena present in electrical systems, such as motors, transformers and lines [2–6], and the motivation towards the development of comprehensive models, seem to point out the requirement for a fractional calculus (FC) approach [7–10].

In an alternative perspective several authors [11–14] have verified that well-known expressions for the electrical potential are related through integer-order integral and derivatives and have proposed its generalization based on the concept of fractional-order poles. Nevertheless, the mathematical generalization towards FC lacks a comprehensive method for its practical implementation.

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Bearing these ideas in mind, we address the analysis and the synthesis of fractional-order multipoles. In Section 2 we start by recalling the method for approximating fractional-order transfer functions based on integer order expressions. In Section 3 we review classical expressions for the static electric potential and we study them in the perspective of FC. Based on this re-evaluation in Section 4 we develop a numerical method for implementing fractional-order electrical potential approximations. Finally, in Section 5 we draw the main conclusions.

2. Approximating fractional order transfer functions

Fractional calculus is a natural extension of the classical mathematics. In fact, since the foundation of the differential calculus the generalization of the concept of derivative and integral to a non-integer order has been the subject of distinct approaches. Due to this reason there are several definitions [15–17] which are proved to be equivalent.

The Laplace definition for a derivative of order $\alpha \in C$ is a direct generalization of the classical integer-order scheme with the multiplication of the signal transform by the s operator yielding (for zero initial conditions):

$$L\{D_{0+}^{\alpha}\varphi\} = s^{\alpha}L\{\varphi\}, \quad Re(\alpha) \geqslant 0. \tag{1}$$

This means that frequency-based analysis methods have a straightforward adaptation to FC. The practical implementation of (1) requires an infinite number of poles and zeros obeying a recursive relationship [18,19]. Nevertheless, in a real approximation the finite number of poles and zeros yields a ripple in the frequency response and a limited bandwidth.

In order to analyze the frequency-based approach to (1) let us consider the recursive circuit represented in Fig. 1 such that:

$$I = \sum_{i=1}^{n} I_i, \quad R_{i+1} = \frac{R_i}{\varepsilon}, \quad C_{i+1} = \frac{C_i}{\eta}, \tag{2}$$

where η and ε are scale factors, I is the current due to an applied voltage V and R_i and C_i are the resistance and capacitance elements of the ith branch of the circuit.

The admittance $Y(\mathrm{j}\omega)$ is given by

$$Y(\mathrm{j}\omega) = \frac{I(\mathrm{j}\omega)}{V(\mathrm{j}\omega)} = \sum_{i=0}^{n} \frac{\mathrm{j}\omega C\varepsilon^{i}}{\mathrm{j}\omega CR + (\eta\varepsilon)^{i}}. \tag{3}$$

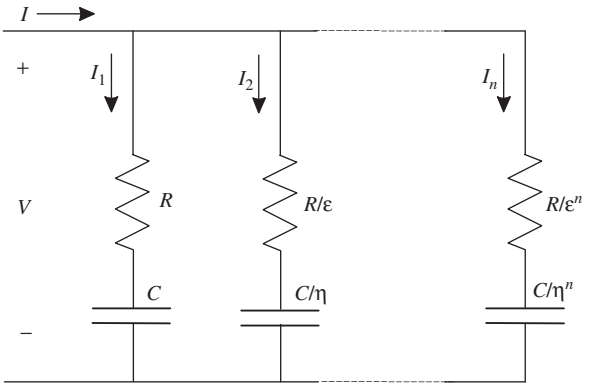

Fig. 1. Electrical circuit with a recursive association of resistance and capacitance elements.

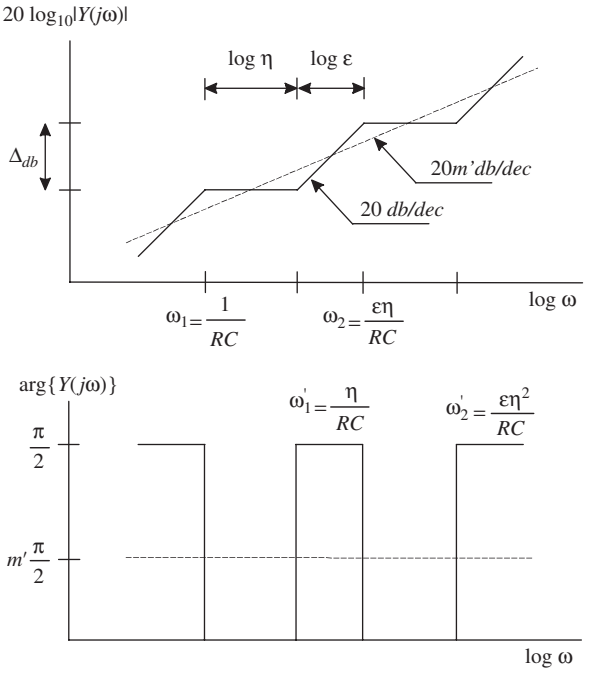

Fig. 2. Bode diagrams of amplitude and phase of $Y(\mathrm{j}\omega)$.

Fig. 2 shows the asymptotic Bode diagrams of amplitude and phase of $Y(\mathrm{j}\omega)$.

The pole and zero frequencies (ω_i and ω_i') obey the recursive relationships:

$$\frac{\omega_{i+1}'}{\omega_i'} = \frac{\omega_{i+1}}{\omega_i} = \varepsilon\eta, \quad \frac{\omega_i}{\omega_i'} = \varepsilon, \quad \frac{\omega_{i+1}'}{\omega_i} = \eta. \tag{4}$$

From the Bode diagram of amplitude or of phase, the average slope m' can be calculated as

$$m' = \frac{\log\varepsilon}{\log\varepsilon + \log\eta}. \tag{5}$$

Consequently, the circuit of Fig. 1 represents an approach to D^{α}, $0<\alpha<1$, with $m'=\alpha$, based on a recursive pole/zero placement in the frequency domain. In fact, this method constitutes the so-called CRONE: Commande Robuste d'Ordre Non Entier, for implementations approximations of fractional order derivatives and integrals.

3. Evaluating classical expressions for the static electric potential

It is well known that, for a homogeneous, linear and isotropic media, the electric potential φ at a point P by a single charge, a dipole and a quadrupole are [20–22]:

$$\varphi = \frac{q}{4\pi\varepsilon_0}\frac{1}{r} + C, \tag{6a}$$

$$\varphi = \frac{ql\cos\theta}{4\pi\varepsilon_0}\frac{1}{r^2} + C, \quad r \gg l, \tag{6b}$$

$$\varphi = \frac{ql^2(3\cos^2\theta - 1)}{4\pi\varepsilon_0}\frac{1}{r^3} + C, \quad r \gg l, \tag{6c}$$

where ε_0 represents the permittivity, q the electric charge, r the radial distance and θ the corresponding angle with the axis.

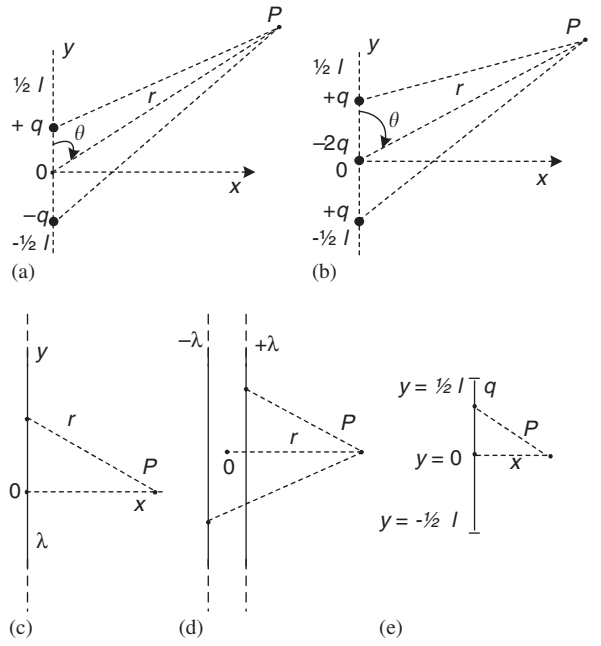

Fig. 3. Electric potential of: (a) dipole; (b) quadrupole; (c) infinite line charge; (d) two opposite charged infinite filaments; (e) straight filament with finite length l and charge q.

The electric potential φ at a point P (Fig. 3) for one very long straight filament carrying a charge λ per unit length, or for two opposite charged filaments are, respectively:

$$\varphi = -\frac{\lambda}{2\pi\varepsilon_0}\ln r + C, \quad C \in \Re, \tag{7a}$$

$$\varphi = \frac{\lambda l\cos\theta}{2\pi\varepsilon_0}\frac{1}{r} + C, \quad r \gg l. \tag{7b}$$

On the other hand, the potential resulting from a planar surface with charge density σ is given by

$$\varphi = -\frac{\sigma}{2\varepsilon_0}r + C, \quad C \in \Re. \tag{8}$$

Analyzing expressions (6a)–(8) we verify the relationship $\Xi : \varphi \sim \{r^{-3}, r^{-2}, r^{-1}, \ln r, r\}$ that corresponds to the application of integer-order derivatives and integrals.

4. On the implementation of fractional order potential

The integer-order differential nature of the potential expressions motivated several authors [11–14] to propose its generalization in a FC perspective. Therefore, a fractional multipole produces at point P a potential $\varphi \sim r^{\alpha}$, $\alpha \in \Re$, where fractional means that we are not restricted to the integer order relationships Ξ observed in the previous section. Nevertheless, besides the abstract manipulation of mathematical expressions, the truth is that there is no practical method, and physic interpretation, for establishing the fractional potential.

Inspired by the integer-order recursive approximation of fractional-order transfer functions presented previously, in this section we develop a numerical method for implementing a fractional order potential.

We start by re-evaluating the potential produced at point $P \equiv (x, y)$ by a straight filament with finite length l and charge q (Fig. 3e):

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{q}{l}\ln\left[\frac{y+\frac{1}{2}l+\sqrt{x^2+(y+\frac{1}{2}l)^2}}{y-\frac{1}{2}l+\sqrt{x^2+(y-\frac{1}{2}l)^2}}\right] + C, \quad C \in \Re. \tag{9}$$

It is well-known that for $x \to \infty$ we have $\varphi \to (q/4\pi\varepsilon_0)(1/x) + C$ and, with $y = 0$, for $x \to 0$ we

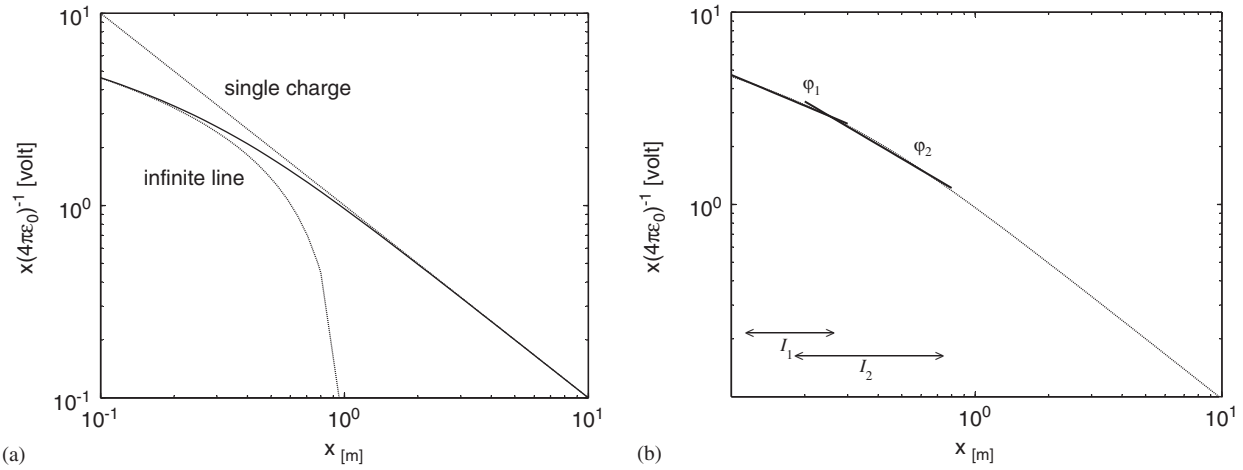

Fig. 4. Comparison of the electric potential φ versus distance x for: (a) a filament with charge $q = 1$, length $l = 1\,\mathrm{m}$, at $y = 0$, a single charge and an infinite line; (b) approximations for I_1: $0.1 < x < 0.3$ and I_2: $0.2 < x < 0.8$.

have $\varphi \to (1/2\pi\varepsilon_0)(q/l)\ln(1/x) + C$. Obviously these limit cases correspond to (6a) and (7a), respectively, that is, to a single charge and to an infinite filament. Fig. 4(a) depicts the potential (9) versus x (with $l = 1\,\mathrm{m}$ and $y = 0$) and, for comparison, the limit cases (6a) and (7a) (for $C = 0$).

In this chart we observe that expression (9) changes smoothly between the two limit cases. Therefore, we can have an intermediate fractional-order relationship as long as we restrict to a limited working range. For example, for the intervals I_1: $0.1 < x < 0.3$ and I_2: $0.2 < x < 0.8$ we get the approximations $\varphi_1 \approx 1.385x^{-0.532}$ and $\varphi_2 \approx 1.031x^{-0.747}$ (Fig. 4(b)), respectively.

This means that standard integer-order potential relationships have a *global* nature while fractional-order potentials have a *local* nature possible to capture only in a restricted region. This conclusion leads to an implementation approach conceptually similar to the one described in Section 2 that is, to an approximation scheme based on a recursive placement of integer-order functions. Nevertheless, in the present case we do not have the analytical formalism of Bode diagrams and, therefore, we decided to adopt a numerical approach.

In this line of thought, we developed a one-dimensional iterative numerical algorithm that places n charges q_i $(i = 0, 1, \ldots, n-1)$ at the symmetrical positions $\pm x_i$ (with exception, for n odd, of $x_0 = 0$ where there is a single charge q_0 that corresponds to the center of the n-array of charges) and compares the resulting approximate potential φ_{app} with the desired reference potential φ_{ref}:

$$\varphi_{\mathrm{app}} = \frac{q_0}{|x|} + \sum_{i=1}^{n-1} \frac{q_i}{4\pi\varepsilon_0}\left(\frac{1}{|x - x_i|} + \frac{1}{|x + x_i|}\right), \quad (10a)$$

$$\varphi_{\mathrm{ref}} = kx^{\alpha}. \quad (10b)$$

The optimization criteria minimizes the square error J yielding:

$$J = \sum_{k=1}^{m} \left(\ln\left|\frac{\varphi_{\mathrm{app}}}{\varphi_{\mathrm{ref}}}\right|\right)^2, \quad (11a)$$

$$\min_i (J), \quad i = 0, 1, \ldots, n-1, \quad (11b)$$

where m is the number of sampling points along the x-axis.

In the present case we consider a log–log perspective, similar to the one used in Bode diagrams, but its modification for a lin–lin case is straightforward. Moreover, in order to reduce the computational load, for an interval $x_A < x < x_B$ we developed a two phase scheme, involving two geometric ratios r_1 and r_2, for capturing the optimal values:

(1) a first phase with a large sampling step $\Delta x = x_A {r_1}^k$ $(k = 0, 1, \ldots)$
(2) second phase with a smaller step $\Delta x = x'_A {r_2}^k$ $(k = 0, 1, \ldots)$ and $r_2 < r_1$ within the previously captured interval $x'_A < x < x'_B$ for evaluating the optimal values with a larger precision.

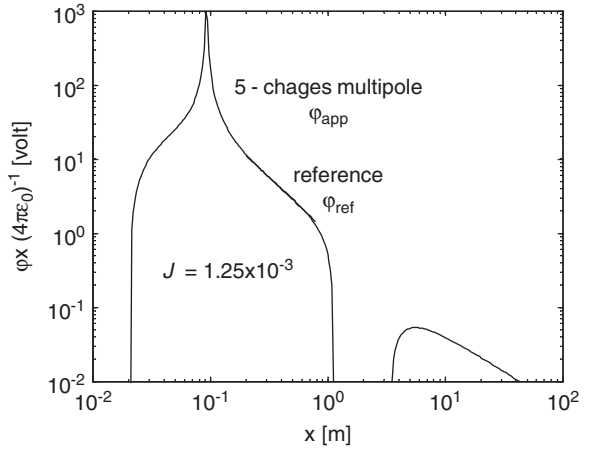

Fig. 5. Comparison of the electric potential φ_{app} and φ_{ref} versus distance x for $\varphi_{ref} = 1.0x^{-1.5}$, $0.2 < x < 0.8$ and a 5-charge approximation.

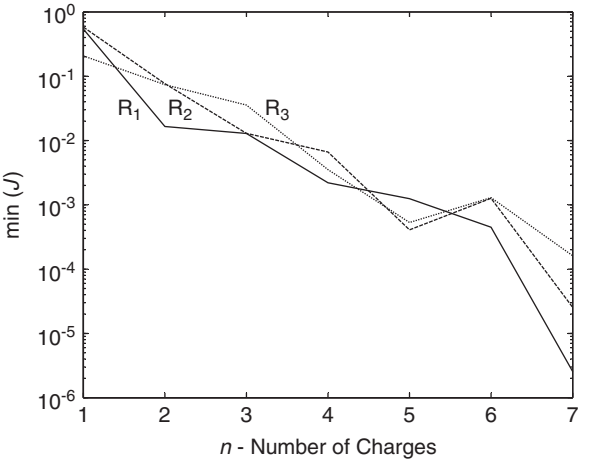

Fig. 6. Approximation error min(J) versus number charges n for $R_1: (r_1, r_2) = (1.3, 1.03)$, $R_2: (r_1, r_2) = (1.4, 1.04)$ and $R_3: (r_1, r_2) = (1.5, 1.05)$, $\varphi_{ref} = 1.0x^{-1.5}$ and $0.2 < x < 0.8$.

For example, Fig. 5 shows a 5-charge approximation for $\varphi_{ref} = 1.0x^{-1.5}$, $0.2 < x < 0.8$, leading to $q_0 = -0.543$, $q_1 = +1.193$ and $q_2 = -0.706$ (with scale factor $\times (4\pi\varepsilon_0)^{-1}$), at $x_0 = 0$, $x_1 = \pm 0.092$ and $x_2 = \pm 1.644$, respectively.

The results show a good fit between the two functions. Nevertheless, for a given application, a superior precision may be required and, in that case, a larger number of charges must be used. In this line of thought, we study the precision of this method for different number of charges, namely for $n = 1$ up to $n = 7$ charges.

Table 1
Charge values and their location for successive iterations $n = \{1, \ldots, 7\}$ when $(r_1, r_2) = (1.3, 1.03)$ and $(r_1, r_2) = (1.4, 1.04)$, $\varphi_{ref} = 1.0\ x^{-1.5}$ and $0.2 < x < 0.8$

	$(r_1, r_2) = (1.3, 1.03)$		$(r_1, r_2) = (1.4, 1.04)$	
n	q_i [C]	x_i [m]	q_i [C]	x_i [m]
1	1.193	0	1.171	0
2	0.706	±0.119	0.837	±0.084
3	−0.543	0	−0.305	0
	0.917	±0.119	0.837	±0.117
4	0.917	±0.092	0.837	±0.117
	−0.706	±1.644	−0.837	±2.420
5	−0.543	0	−0.427	0
	1.193	±0.092	1.171	±0.084
	−0.706	±1.644	−0.837	±1.729
6	1.193	±0.092	1.171	±0.084
	−0.19 0	±0.119	−0.111	±0.117
	−0.917	±1.644	−1.171	±1.729
7	1.193	0	1.171	0
	0.418	±0.119	0.427	±0.1171
	−1.193	±2.778	−1.171	±2.42 0
	−0.706	±3.612	−0.305	±2.42 0

Fig. 6 depicts min(J) versus n, for $R_1: (r_1, r_2) = (1.3, 1.03)$, $R_2: (r_1, r_2) = (1.4, 1.04)$ and $R_3: (r_1, r_2) = (1.5, 1.05)$, and confirms that we have a better precision the larger the number of charges and the smaller the r_1. This chart can be approximated closely by the following expressions $\min(J) \approx 1.740e^{-1.655n}$, $\min(J) \approx 1.832e^{-1.492n}$ and $\min(J) \approx 0.716e^{-1.205n}$, respectively.

Table 1 shows the charges values and the corresponding positions, for $(r_1, r_2) = (1.3, 1.03)$ and $(r_1, r_2) = (1.4, 1.04)$ with $\varphi_{ref} = 1.0x^{-1.5}, 0.2 < x < 0.8$.

We verify that the position of the charges varies significantly with the precision of the algorithm, namely with the increment r_1 of the numerical grid. Therefore, the pattern revealed by the charge is not clear and its comparison with a fractal recursive layout is still under investigation.

The experiments also reveal that it is possible to find more than one 'good' solution. For example, Fig. 7 shows a 5-charge approximation for $\varphi_{ref} = 1.0x^{-1.5}$, $0.2 < x < 0.8$, leading to $q_0 = +0.039$, $q_1 = +0.113$ and $q_2 = +0.543$ (with scale factor $\times (4\pi\varepsilon_0)^{-1}$), at $x_0 = 0, x_1 = \pm 0.155$ and $x_2 = \pm 0.119$, respectively.

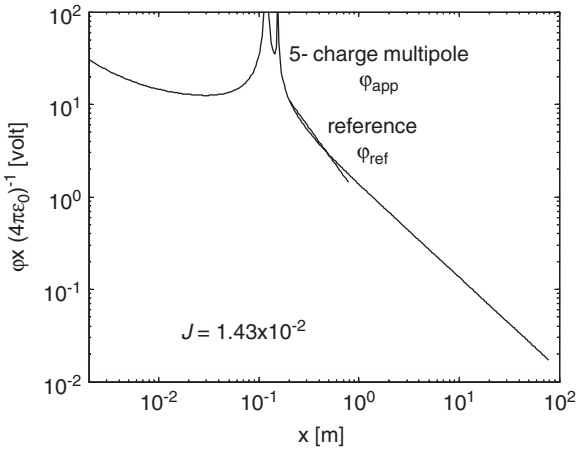

Fig. 7. Comparison of the electric potential φ_{app} and φ_{ref} versus distance x for $\varphi_{ref} = 1.0x^{-1.5}$, $0.2 < x < 0.8$ and a 5-charge approximation.

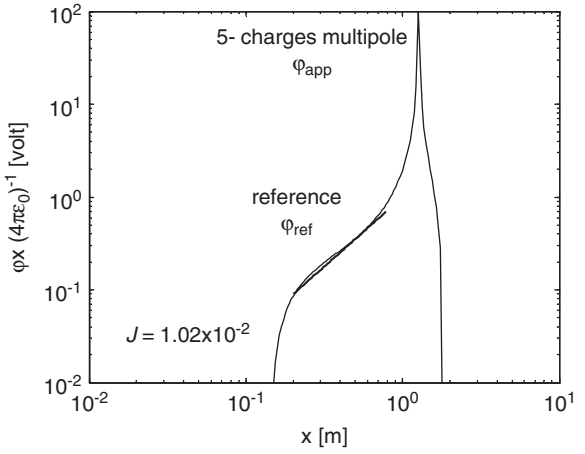

Fig. 8. Comparison of the electric potential φ_{app} and φ_{ref} versus distance x for $\varphi_{ref} = 1.0x^{1.5}$, $0.2 < x < 0.8$ and a 5-charge approximation.

On the other hand, with this method it is also possible to have a reference potential with positive slope. Fig. 8 shows a 5-charge approximation for $\varphi_{ref} = 1.0x^{1.5}$, $0.2 < x < 0.8$, leading to $q_0 = -0.039$, $q_1 = +0.706$ and $q_2 = -1.193$ (with scale factor $\times(4\pi\varepsilon_0)^{-1}$) at $x_0 = 0$, $x_1 = \pm 1.265$ and $x_2 = \pm 2.778$, respectively.

Table 2 shows the charges values and the corresponding positions, for $(r_1, r_2) = (1.3, 1.03)$, $\varphi_{ref} = 1.0x^{1.5}$ and $0.2 < x < 0.8$.

Fig. 9(a) depicts approximation error min (J) versus n for different size of interval approximations, I: $0.5 < x < 1.0$, I': $0.5 < x < 1.5$, and I'': $0.5 < x < 2.0$, $\varphi_{ref} = 1.0x^{-1.5}$ and $(r_1, r_2) = (1.4, 1.04)$. We observe that the error min(J) decreases when reducting the size of interval approximation.

Table 2
Charge values and their location for successive iterations $n = \{1, \ldots, 6\}$ when $(r_1, r_2) = (1.3, 1.03)$, $\varphi_{ref} = 1.0\ x^{1.5}$ and $0.2 < x < 0.8$

$(r_1, r_2) = (1.3, 1.03)$		
n	q_i [C]	x_i [m]
1	0.067	0
2	0.113	±1.265
3	−0.087	0
	0.543	±2.137
4	0.418	±1.265
	−1.193	±4.696
5	−0.039	0
	0.706	±1.265
	−1.193	±2.779
6	0.917	±1.265
	−1.193	±2.778
	−1.193	±4.695

The present numerical algorithm evaluates a multi-grid of possible values for x_i and q_i. Several heuristics were implemented to avoid unnecessary computations, namely when a given iteration exceeds previous optimal trials. Nevertheless, the computational time T increases with the number of charges n and the adopted grid. Fig. 9(b) shows T versus n for $(r_1, r_2) = (1.4, 1.04)$, $\varphi_{ref} = 1.0x^{-1.5}$ and $0.2 < x < 0.8$. The chart reveals an exponential variation $T \approx 2.422 \times 10^{-6} e^{2.894n}$, for $n \geqslant 3$. Therefore, approximations with an high number of charges require an high computational time. In order to overcome this problem, a genetic algorithm is presently under development.

5. Conclusions

This paper addressed the problem of implementing a fractional-order electric potential. It was adopted an algorithm inspired on the Bode diagram recursive scheme. While in the Bode diagrams both numerical and analytical approaches are possible, in the present case only a numerical evaluation was implemented and the analytical counterpart remains to be investigated. In fact, this paper constitutes a

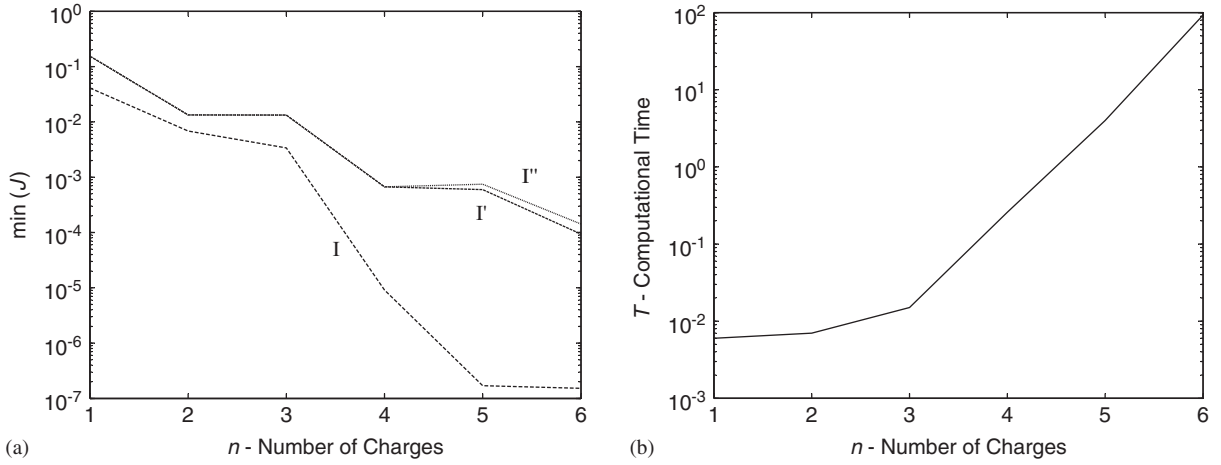

Fig. 9. (a) Approximation error min(J) versus number charges n for I: $0.5<x<1.0$, I': $0.5<x<1.5$, and I'': $0.5<x<2.0$, $\varphi_{\text{ref}} = 1.0x^{-1.5}$ and $(r_1, r_2) = (1.4, 1.04)$; (b) comparison of computational time T versus number of charges n, for $(r_1, r_2) = (1.4, 1.04)$, $\varphi_{\text{ref}} = 1.0x^{-1.5}$ and $0.2<x<0.8$.

first step towards the development of a systematic design technique and, consequently, several other aspects must be evaluated. Research on the approximation feasibility and convergence, error variation with the range and the number of charges, improvement when adopting an extended library of primitives rather than, merely, point charges and its extension to the three-dimensional space is presently under development.

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