ELECTRICAL SKIN PHENOMENA: A FRACTIONAL CALCULUS ANALYSIS

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Abstract

The internal impedance of a wire is the function of the frequency. In a conductor, where the conductivity is sufficiently high, the displacement current density can be neglected. In this case, the conduction current density is given by the product of the electric field and the conductance. One of the aspects of the high-frequency effects is the skin effect (*SE*). The fundamental problem with *SE* is it attenuates the higher frequency components of a signal.

The *SE* was first verified by Kelvin in 1887. Since then many researchers developed work on the subject and presently a comprehensive physical model, based on the Maxwell equations, is well established.

The Maxwell formalism plays a fundamental role in the electromagnetic theory. These equations lead to the derivation of mathematical descriptions useful in many applications in physics and engineering. Maxwell is generally regarded as the 19th century scientist who had the greatest influence on 20th century physics, making contributions to the fundamental models of nature.

The Maxwell equations involve only the integer-order calculus and, therefore, it is natural that the resulting classical models adopted in electrical engineering reflect this perspective. Recently, a closer look of some phenomenas present in electrical systems and the motivation towards the development of precise models, seem to point out the requirement for a fractional calculus approach. Bearing these ideas in mind, in this study we address the *SE* and we re-evaluate the results demonstrating its fractional-order nature.

Keywords

Skin effect, eddy currents, electromagnetism, fractional calculus.

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1 Introduction

Some experimentation with magnets was beginning in the late 19th century. By then reliable batteries had been developed and the electric current was recognized as a stream of charge particles. Maxwell developed a set of equations expressing the basic laws of electricity and magnetism, and he demonstrated that these two phenomenas are complementary aspects of electromagnetism. He showed that electric and magnetic fields travel through space, in the form of waves, at a constant velocity.

The skin effect (*SE*) is one subject who can be explained by the Maxwell's equations. The *SE* is the tendency of a high-frequency electric current to distribute itself in a conductor so that the current density near the surface is greater than that at its core. This phenomenon increases the effective resistance of the conductor with the frequency of the current. The effect is most pronounced in radio-frequency systems, especially antennas and transmission lines [1], but it can also affect the performance of high-fidelity sound equipment, by causing attenuation in the treble range. The first study of *SE* was explained by Lord Kelvin in 1887, but many other scientists had made contributions to improve the comprehension of this theme.

The *SE* can be reduced by using stranded rather than solid wire. This increases the effective surface area of the wire for a given wire gauge. It is simple to see that the spatial variation of the fields in vacuum is much smaller than the special variation in the metal. Therefore, in usual study, for the purposes of evaluating the fields in the conductor, the spatial variation from the wave length outside the conductor can be ignored. For the usual case the radii of curvature of the surface should be much larger than a skin depth, the solution is straightforward. To analyse this phenomenon, we apply the Maxwell's equations that relate the solutions for these fields. More often, however, some of the parameters that tend to be considered are the capacitance per length, inductance per length, and their relationship with the signals, the nominal propagation velocity, and the characteristic impedance of the system.

In our study we apply the Bessel functions to compute values of cable impedance Z. For the sake of clarity we plot some values of the low and high-frequency approximations of impedance. We verify the fractional order of these systems, namely the half-order nature of dynamic phenomenon.

Having these ideas in mind this paper is organized as follows. Section 2 summarizes the mathematical description of the *SE*. Section 3 re-evaluates the *SE* demonstrating its fractional-order dynamics. After clarifying the fundamental concepts, section 4 addresses the case of eddy (or Foucault) currents that occur in electrical machines. Finally, section 5 draws the main conclusions.

2 The Skin Effect

In the differential form the Maxwell equations are [2]:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1a}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{1b}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{1c}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1d}$$

where **E**, **D**, **H**, **B**, and **J** are the vectors of electric field intensity, electric flux density (or electric displacement), magnetic field intensity, magnetic flux density and the current density, respectively, and ρ and t are the charge density and time. Moreover, for a homogeneous, linear, and isotropic media, we can establish the relationships:

$$\mathbf{D} = \varepsilon \mathbf{E} \tag{2a}$$

$$\mathbf{B} = \mu \mathbf{H} \tag{2b}$$

$$\mathbf{J} = \sigma \mathbf{E} \tag{2c}$$

where ε, μ, and σ are the electrical permittivity, the magnetic permeability and the conductivity, respectively.

In order to study the *SE* we start by considering a cylindrical conductor with radius r_0 conducting a current I along its longitudinal axis. In a conductor, even for high frequencies, the term $\partial \mathbf{D}/\partial t$ is negligible in comparison with the conduction term **J** or, by other words, the displacement current is much lower than the conduction current. Therefore, for a radial distance $r < r_0$ the application of the Maxwell's equations with the simplification of (1b) leads to the expression [3, 4]:

$$\frac{\partial^2 E}{\partial r^2} + \frac{1}{r}\frac{\partial E}{\partial r} = \sigma\mu \frac{\partial E}{\partial t} \tag{3}$$

For a sinusoidal field we can adopt the complex notation $E = \sqrt{2}\tilde{E}e^{i\omega t}$, where $i = \sqrt{-1}$, yielding:

$$\frac{d^2 \tilde{E}}{dr^2} + \frac{1}{r}\frac{d\tilde{E}}{dr} + q^2 \tilde{E} = 0 \tag{4}$$

with $q^2 = -i\omega\sigma\mu$.

Equation (4) is a particular case of the Bessel equation that, for the case under study, has solution of the type:

$$\tilde{E} = \frac{q}{2\pi r_0 \sigma}\frac{J_0\left(qr\right)}{J_1\left(qr_0\right)} I,\ 0 \le r \le r_0 \tag{5}$$

where J_0 and J_1 are complex valued Bessel functions of the first kind of orders 0 and 1, respectively.

Equation (5) establishes the so-called *SE* that consists on having a non-uniform current density, namely a low density near the conductor axis and an high density on surface, the higher the frequency ω.

An important measure of the *SE* is the so-called skin depth $\delta = \left(\frac{2}{\omega\mu\sigma}\right)^{1/2}$, corresponding to the distance δ below the conductor surface, for which the field reduces to e^{-1} of its value.

The total voltage drop is $\tilde{Z}\tilde{I} = \tilde{E}\tilde{I}$ that, for a conductor of length l_0, results:

$$\tilde{Z} = \tilde{E} = \frac{ql_0}{2\pi r_0\sigma}\frac{J_0\left(qr_0\right)}{J_1\left(qr_0\right)} \tag{6}$$

where $\tilde{Z}$ is the equivalent electrical complex impedance.

Knowing [5] the Taylor series:

$$J_0\left(x\right) = 1 - \frac{x^2}{2^2} + \cdots, \quad J_1\left(x\right) = \frac{x}{2} - \frac{x^3}{2^2 4} + \cdots \tag{7}$$

and, for large values of x, the asymptotic expansion:

$$J_n\left(x\right) = \sqrt{\frac{2}{\pi x}}\cos\left(x - n\frac{\pi}{2} - \frac{\pi}{4}\right), \ n = 0, 1, \cdots \tag{8}$$

we can obtain the low and high-frequency approximations of $\tilde{Z}$:

$$\omega \to 0 \ \Rightarrow \ \tilde{Z} \approx \frac{l_0}{\pi r_0^2\sigma} \tag{9a}$$

$$\omega \to \infty \ \Rightarrow \ \tilde{Z} \approx \frac{l_0}{2\pi r_0}\sqrt{\frac{\omega\mu}{2\sigma}}\left(1 + i\right) \tag{9b}$$

In the classical *SE* the mean free path l that the electrons can travel between subsequent scattering events is less than the skin depth δ. Therefore, for $\delta >> l$ we have a local relation and the value of $\mathbf{J}$ at a given point is determined simply by the value of $\mathbf{E}$ at that point. The Ohm's law (2c) is valid, the normal *SE* yields $\delta \sim \omega^{-1/2}$, and the impedance $Z = R + iX$ such that $R = X \sim \omega^{1/2}$.

For very low temperatures the *SE* behaves somewhat differently. In the anomalous skin effect (*ASE*) $\delta << l$ the relation between $\mathbf{J}$ and $\mathbf{E}$ is non-local and the electrons are subjected to the field for only a part of its transit time between two collisions between the metal ions [12, 13]. Consequently, it is equivalent to a smaller electron concentration in the skin layer and that causes a poorer conductivity. The anomalous skin depth yields $\delta \sim \omega^{-1/3}$, and the impedance $Z = R + iX$ is such that $R = X\big/\sqrt{3} \sim \omega^{2/3}$.

In this paper we will focus only on the *SE* but the extension of the proposed methods to the *ASE* is straightforward.

3 The Eddy Currents

The previous physical concepts and mathematical tools can be adopted in more complex systems. The "Eddy Currents" phenomenon common in electrical machines, such as transformers and motors, can be modelled using an identical approach.

Let us consider the magnetic circuit of an electrical machine constituted by a laminated iron core. Each ferromagnetic metal sheet with permeability μ has thickness d and width b ($b \gg d$) making a closed magnetic circuit with an average length l_0. The total pack of ferromagnetic metal sheet make a height a while embracing a coil having n turns with current I.

The contribution of the magnetic core to the coil impedance is (for details see [3]):

$$\tilde{Z} = \frac{2\mu ab\, j\omega\, n^2}{(1+i)\,\beta L d} \tanh\left[(1+i)\,\beta \frac{d}{2}\right] \tag{10}$$

where $\beta = \sqrt{\omega\sigma\mu/2}$.

Alternatively, expression (12) can be re-written as:

$$\tilde{Z} = \frac{\mu ab\, n^2}{l_0}\omega \cdot \frac{[\sinh(\beta d) - \sin(\beta d)] + i\,[\sinh(\beta d) + \sin(\beta d)]}{(\beta d)\,[\cosh(\beta d) + \cos(\beta d)]} \tag{11}$$

We can obtain the low and high-frequency approximations of $\tilde{Z}$:

$$\omega \to 0 \;\Rightarrow\; \tilde{Z} \approx i\omega \frac{\mu ab\, n^2}{l_0} \tag{12a}$$

$$\omega \to \infty \;\Rightarrow\; \tilde{Z} \approx \frac{\mu ab\, n^2}{l_0}\frac{1}{d}\sqrt{\frac{2\omega}{\sigma\mu}}\,(1+i) \tag{12b}$$

Once more we have a clear half-order dependence of $\tilde{Z}$ (i.e., $\tilde{Z} \sim \omega^{1/2}$) while the standard approach is to assign frequency-dependent "equivalent" resistance R and inductance L given by $R + i\omega L = \tilde{Z}$.

4 A Fractional Calculus Perspective

In this section we re-evaluate the expressions obtained for the SE and the Eddy phenomena, in the perspective of fractional calculus.

In the SE, to avoid the complexity of the transcendental Eq. (6), the standard approach in electrical engineering is to assign a resistance R and inductance L given by $R + i\omega L = \tilde{Z}$. Nevertheless, although widely used, this method is clearly inadequate because the model parameter values $\{R, L\}$ vary with the frequency. Moreover, (9b) points out the half-order nature of the dynamic phenomenon, at high frequencies (i.e., $\tilde{Z} \sim \omega^{1/2}$), which is not captured by and integer-order approach. A possible approach that eliminates those problems is to adopt the fractional calculus [6, 7, 8, 9, 10]. Joining the two asymptotic expressions (9) we can establish several types of approximations [11], namely the two expressions:

$$\tilde{Z}_{a1} \approx \frac{l_0}{\pi r_0^2 \sigma} \left[i\omega \left(\frac{r_0}{2} \right)^2 \mu\sigma + 1 \right]^{1/2} \tag{13a}$$

$$\tilde{Z}_{a2} \approx \frac{l_0}{\pi r_0^2 \sigma} \left\{ \left[i\omega \left(\frac{r_0}{2} \right)^2 \mu\sigma \right]^{1/2} + 1 \right\} \tag{13b}$$

In order to analyse the feasibility of (13) we define the polar, amplitude, and phase relative errors as:

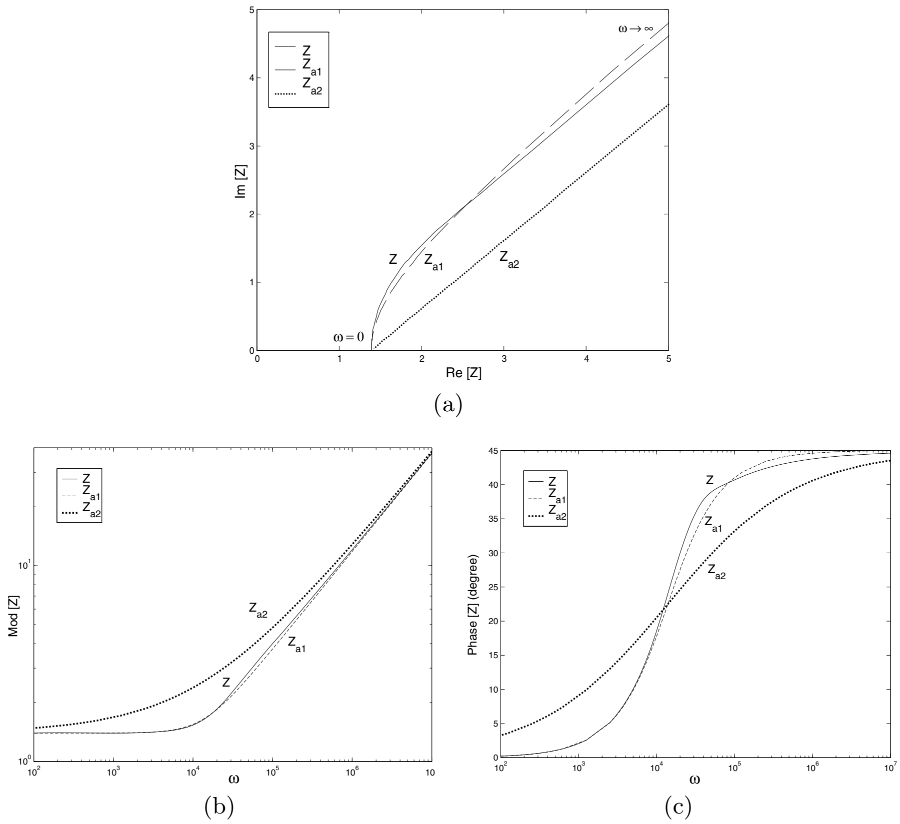

Fig. 1. Diagrams of the theoretical electrical impedance $\tilde{Z}(i\omega)$ and the two approximate expressions $\tilde{Z_{a1}}$, $\tilde{Z_{a2}}$ (10) with: $\sigma = 5.7\ 10^7 \Omega^{-1}$ m, $l_0 = 10^3$ m, $r_0 = 2.0\, 10^{-3}$ m, $\mu = 1.257\, 10^{-6}$ Hm^{-1} (a) Polar, (b) Bode amplitude, and (c) Bode phase.

$$\varepsilon_{Rk}(\omega) = (\tilde{Z} - \tilde{Z}_{ak})/\left|\tilde{Z}\right| \tag{14a}$$

$$\varepsilon_{Mk} = Mod\ \{\varepsilon_{Rk}(\omega)\} \tag{14b}$$

$$\varepsilon_{\phi k} = Phase\ \{\varepsilon_{Rk}(\omega)\} \tag{14c}$$

where the index $k = \{1, 2\}$ represents the two types of approximation.

Figure 1 compares the polar and Bode diagrams of amplitude and phase for expressions (6) and (13) revealing a very good fit in the two cases. On the other hand, Fig. 2 depicts the errors in the charts of polar, amplitude, and phase relative errors, respectively. These figures reveal that the results obtained with the expression (13a) have an better approximation than Eq. (13b), that presents an larger error in the middle of the frequency range.

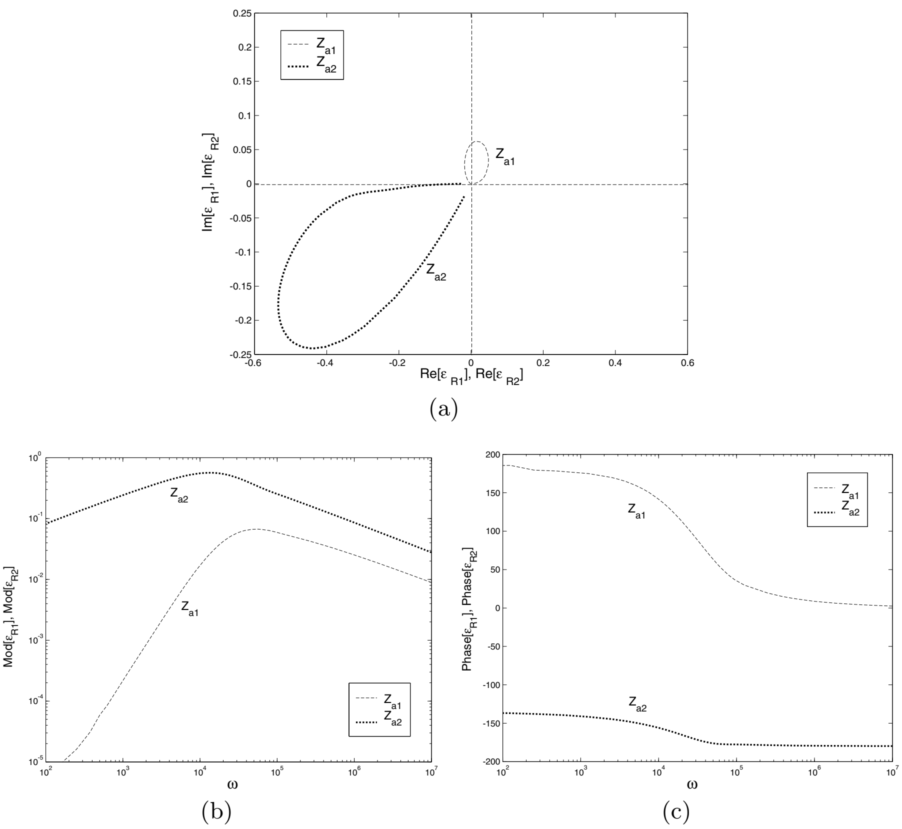

Fig. 2. (a) Polar, (b) amplitude, and (c) phase relative errors for the two approximate expressions $\tilde{Z_{a1}}$ and $\tilde{Z_{a2}}$.

Now we re-evaluate also expressions (10) having in mind the tools of fractional calculus.

A possible approach that avoids the problems posed by the transcendental expression (10) is to joint the two asymptotic expressions (12). Therefore, we can establish several types of approximations, namely the two fractions:

$$\tilde{Z}_{a1} \approx \frac{i\omega\ \mu ab\ n^2}{l_0}\left[i\omega\left(\frac{d}{2}\right)^2 \mu\sigma + 1\right]^{-1/2} \tag{15a}$$

$$\tilde{Z}_{a2} \approx \frac{i\omega\ \mu ab\ n^2}{l_0}\left\{\left[i\omega\left(\frac{d}{2}\right)^2 \mu\sigma\right]^{-1/2} + 1\right\} \tag{15b}$$

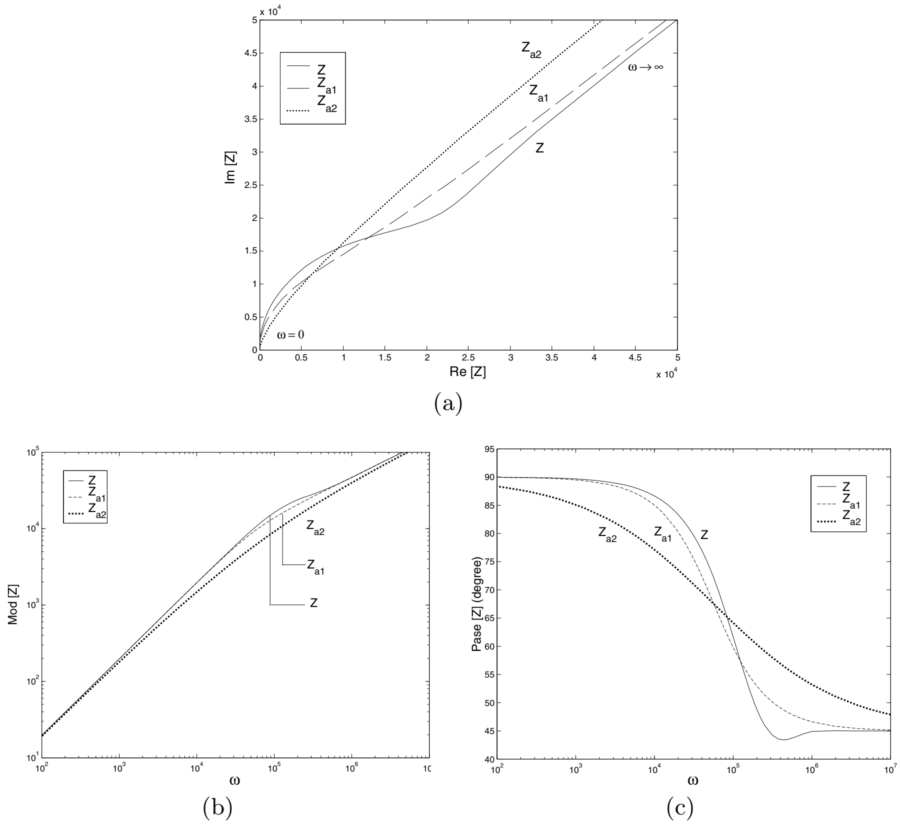

Fig. 3. Diagrams of the theoretical electrical impedance $\tilde{Z}(i\omega)$ and the two approximate expressions $\tilde{Z_{a1}}$ and $\tilde{Z_{a2}}$ (15), with: $l_0 = 1.0$ m, $a = 0.28$ m, $b = 0.28$ m, $d = 2.0\ 10^{-3}$ m, $n = 100$, $\sigma = 7.0\ 10^4 \Omega^{-1}$m, $\mu = 200 \cdot 1.257\ 10^{-6}$ Hm^{-1} (a) polar, (b) Bode amplitude, and (c) Bode phase.

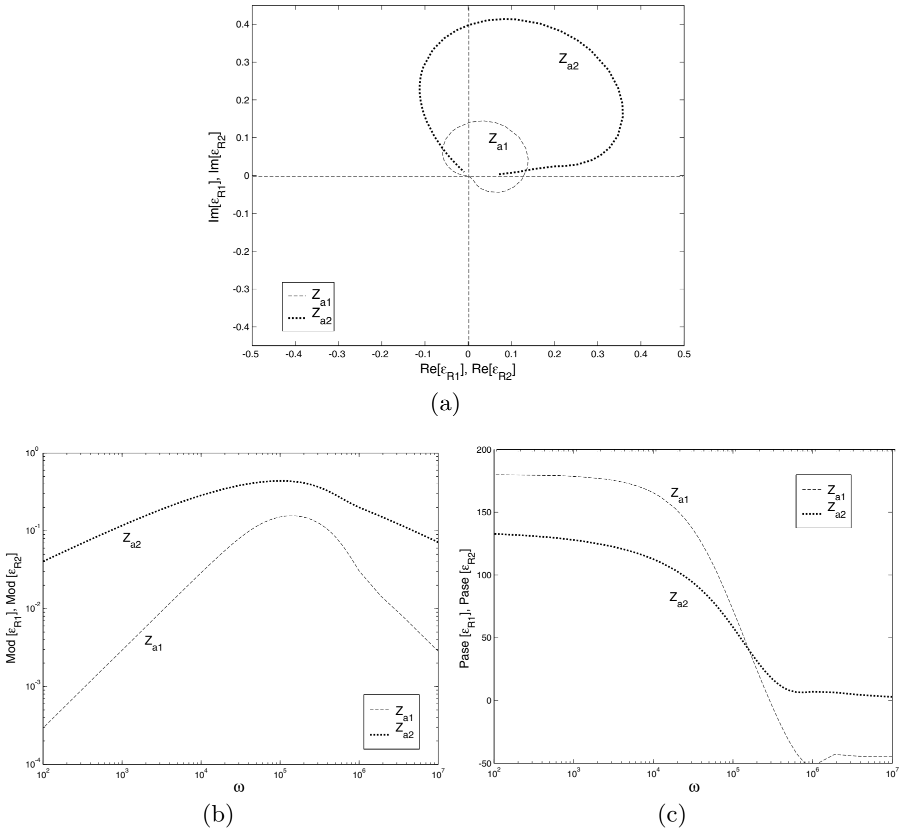

Fig. 4. (a) Polar, (b) amplitude, and (c) phase relative errors for the two approximate expressions $\tilde{Z_{a1}}$ and $\tilde{Z_{a2}}$.

Figure 3 compares the polar and Bode diagrams of amplitude and phase for expressions (10) and (15) revealing a very good fit in the two cases. Figure 4 depicts the relative errors in the charts of polar, amplitude, and phase, respectively. These figures, reveal that the results obtained with expression (15a) have an better approximation, comparatively with Eq. (15b), that presents larger errors in the middle of the frequency range.

5 Conclusions

The classical electromagnetism and the Maxwell equations involve integer-order derivatives, but lead to models requiring a fractional calculus perspective to be fully interpreted. Another aspect of interest is that in all cases we get half-order models. Recent results point out that this is due to the particular geometry of the addressed problems. Therefore, the analysis of different conductor geometries and its relationship with distinct values of the fractional-order model is under development.

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