Fractional Dynamics and Control of Distributed Parameter Systems

J. A. Tenreiro Machado[1], Isabel S. Jesus[1], J. Boaventura Cunha[2]
and József K. Tar[3]

[1]Institute of Engineering,
Polytechnic Institute of Porto, Porto, Portugal
{jtm,isj}@isep.ipp.pt

[2]University of Trás-os-Montes and Alto Douro,
Institute of Intelligent Engineering Systems, Vila-Real, Portugal
jboavent@utad.pt

[3]Institute of Intelligent Engineering Systems,
Budapest Tech, John von Neumann Faculty of Informatics, Budapest, Hungary
tar@nik.bmf.hu

Abstract — *Fractional Calculus (FC) goes back to the beginning of the theory of differential calculus. Nevertheless, the application of FC just emerged in the last two decades, due to the progress in the area of chaos that revealed subtle relationships with the FC concepts. In the field of dynamical systems theory some work has been carried out but the proposed models and algorithms are still in a preliminary stage of establishment. Having these ideas in mind, the paper discusses a FC perspective in the study of the dynamics and control of some distributed parameter systems.*

1   Introduction

The generalization of the concept of derivative $D^\alpha[f(x)]$ to non-integer values of α goes back to the beginning of the theory of differential calculus. In fact, Leibniz, in his correspondence with Bernoulli, L'Hôpital and Wallis (1695), had several notes about the calculation of $D^{1/2}[f(x)]$. Nevertheless, the development of the theory of Fractional Calculus (*FC*) is due to the contributions of many mathematicians such as Euler, Liouville, Riemann and Letnikov [1, 2, 3]. The adoption of the *FC* in control algorithms has been recently studied using the frequency and discrete-time domains [4, 5]. Nevertheless, this research is still giving its first steps and further investigation is required.

This article studies the dynamics and control of classical distributed parameter linear systems. In this perspective, the paper is organized as follows. Section 2 presents the main mathematical aspects of the theory of *FC*. Section 3 analyzes the dynamics of partial differential equations, corresponding to electrical transmission lines and to heat diffusion systems, on the perspective of *FC*. Finally, section 4 draws the main conclusions.

2 Theory of Fractional Calculus

The mathematical definition of a derivative of fractional order has been the subject of several different approaches [1]. For example, (1) and (2), represent the Laplace (for zero initial conditions) and the Grünwald-Letnikov definitions of the fractional derivative of order α of the signal $x(t)$:

$$D^\alpha\left[x(t)\right] = L^{-1}\left\{s^\alpha X(s)\right\} \tag{1}$$

$$D^\alpha x\left(t\right) = \lim_{h\to 0}\left[\frac{1}{h^\alpha}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma\left(\alpha+1\right)}{\Gamma\left(k+1\right)\Gamma\left(\alpha-k+1\right)}x\left(t-kh\right)\right] \tag{2}$$

where Γ is the gamma function and h is the time increment.

In the literature were proposed two methods for implementing fractional-order derivatives, namely the frequency-based and the discrete-time approaches.

In order to analyze a frequency-based approach to D^α, $0<\alpha<1$, let us consider the recursive circuit represented on Figure 1 such that:

$$I = \sum_{i=0}^{n}I_i,\quad R_{i+1}=\frac{R_i}{\varepsilon},\quad C_{i+1}=\frac{C_i}{\eta} \tag{3}$$

where η and ε are scale factors, I is the current due to an applied voltage V and R_i and C_i are the resistance and capacitance elements of the ith branch of the circuit.

The admittance $Y(j\omega)$ is given by:

$$Y\left(j\omega\right) = \frac{I\left(j\omega\right)}{V\left(j\omega\right)} = \sum_{i=0}^{n}\frac{j\omega C\varepsilon^i}{j\omega CR + \left(\eta\varepsilon\right)^i} \tag{4}$$

Figure 2 shows the asymptotic Bode diagram of amplitude of $Y(j\omega)$. The pole and zero frequencies (ω_i and $\omega_i^{'}$) obey the recursive relationships:

$$\frac{\omega_{i+1}'}{\omega_i'} = \frac{\omega_{i+1}}{\omega_i} = \varepsilon\eta,\quad \frac{\omega_i}{\omega_i'} = \varepsilon,\quad \frac{\omega_{i+1}'}{\omega_i} = \eta \tag{5}$$

From the Bode diagram of amplitude or of phase, the average slope m' can be calculated as:

$$m' = \frac{\log\varepsilon}{\log\varepsilon + \log\eta} \tag{6}$$

Consequently, the circuit of Figure 1 represents an approach to D^α, $0<\alpha<1$, with $m' = \alpha$, based on a recursive pole/zero placement in the frequency domain.

As mentioned in (1) the Laplace definition for a derivative of order $\alpha \in$ C is a 'direct' generalization of the classical integer-order scheme with the multiplication of the signal transform by the s operator. Therefore, in what concerns automatic control theory this means that frequency-based analysis methods have a straightforward adaptation to their fractional-order counterparts.
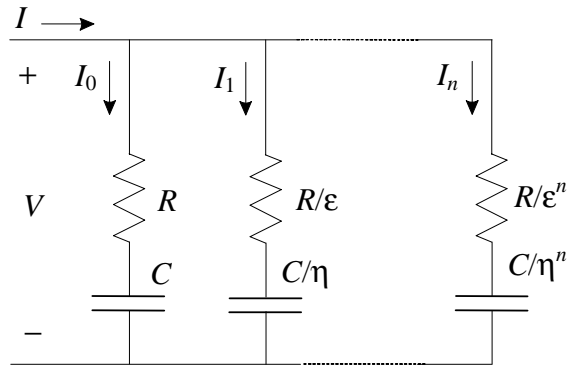
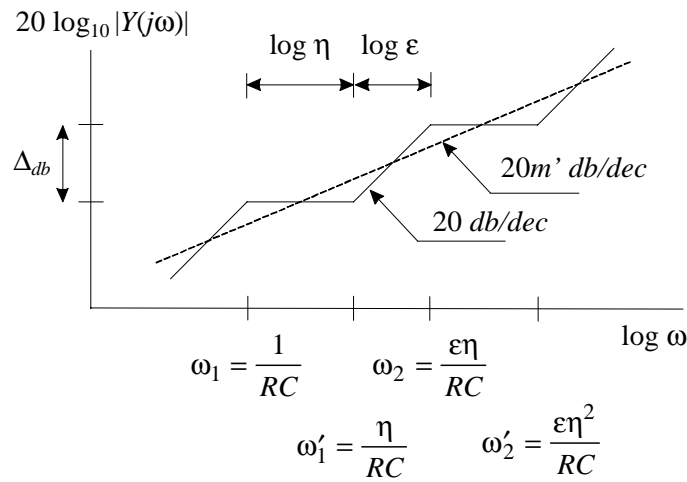Figure 1: Electrical circuit with a recursive association of resistances and capacitances.

Figure 2: Bode diagrams of amplitude of $Y(j\omega)$.

Nevertheless, the implementation based on the Laplace definition (adopting the frequency domain) requires an infinite number of poles and zeros obeying a recursive relationship [4]. In a real approximation the finite number of poles and zeros yields a ripple in the frequency response and a limited bandwidth.

The Grünwald-Letnikov formulation [5] inspired a discrete-time calculation algorithm, based on the approximation of the time increment h through the sampling period T, yielding the equation in the z domain:

$$\frac{Z\left\{D^\alpha x\left(t\right)\right\}}{X\left(z\right)} \approx \frac{1}{T^\alpha} \sum_{k=0}^{r} \frac{\left(-1\right)^k \Gamma\left(\alpha+1\right)}{k! \; \Gamma\left(\alpha-k+1\right)} z^{-k} = \left(\frac{1-z^{-1}}{T}\right)^\alpha \tag{7}$$

An implementation of (7) corresponds either to a r-term truncated series or to a Padé fraction.

An important aspect of fractional-order controllers [4] can be illustrated through the elemental control system represented in Figure 3, with open-loop transfer function $G(s) = Ks^{-\alpha}$ $(1<\alpha<2)$ in the forward path. The open-loop Bode diagrams (Figure 4) of amplitude and phase have a slope of 20α *dB/dec* and a constant phase of $-\alpha\pi/2 rad$, respectively. Therefore, the closed-loop system has a constant phase margin of
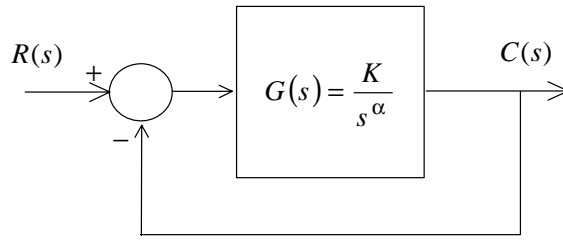
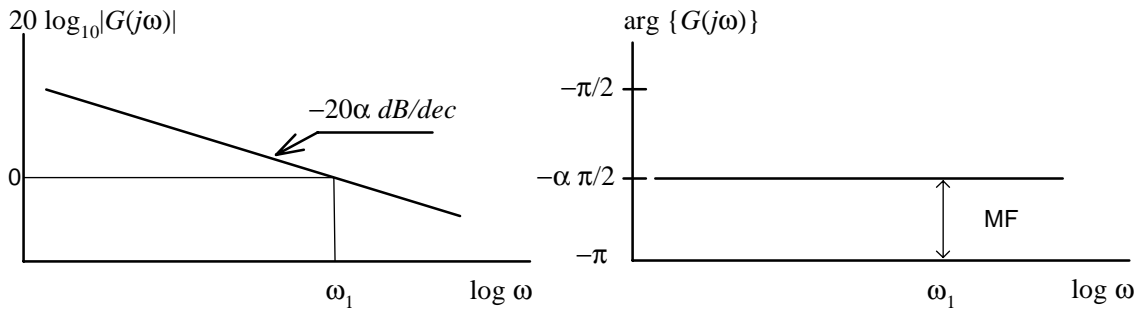Figure 3: Block diagram for an elemental feedback control system of fractional order α.

Figure 4: Bode diagrams of $G(j\omega)$ for $1 < \alpha < 2$.

$MF = \pi\left(1 - \alpha/2\right) rad$ that is independent of the system gain K.

3 Distributed Parameter Systems

In this section we analyze the fractional order dynamics of two distributed parameter systems namely the electrical transmission lines and the heat diffusion.

3.1 Electrical Transmission Lines

The differential equations for a uniform transmission line [6, 7, 8] are found by considering an infinitesimal length *dx* located at coordinate x. This line section has series inductance and resistance *Ldx* and *Rdx* and shunt conductance and capacitance *Gdx* and *Cdx* as depicted in Figure 5.

The application of the Kirchoff's laws to the circuit leads to the set of *PDE*s:

$$\frac{\partial v(x,t)}{\partial x} = -\frac{L\partial i(x,t)}{\partial t} - Ri(x,t) \tag{8a}$$

$$\frac{\partial i(x,t)}{\partial x} = -\frac{C\partial v(x,t)}{\partial t} - Gv(x,t) \tag{8b}$$

where t represent time, v voltage and i electrical current.

A few simple calculations allow us to eliminate one variable and to explicit the differential equation either to v or to i, yielding:

$$\frac{\partial^2 v(x,t)}{\partial x^2} = LC\frac{\partial^2 v(x,t)}{\partial t^2} + (LG + RC)\frac{\partial v(x,t)}{\partial t} + RGv(x,t) \tag{9a}$$
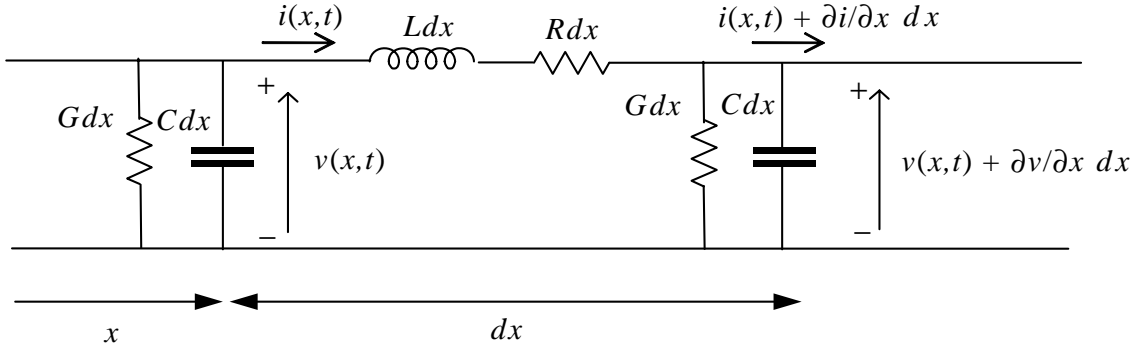
298

Figure 5: Electrical circuit of an infinitesimal portion of a uniform transmission line.

$$\frac{\partial^2 i(x,t)}{\partial x^2} = LC\frac{\partial^2 i(x,t)}{\partial t^2} + (LG + RC)\frac{\partial i(x,t)}{\partial t} + RGi(x,t) \tag{9b}$$

It is interesting to note that when $L = 0$ and $G = 0$ equations (9) reduces to the equivalent of the heat diffusion equation (see section 3.2), where v and i are the analogs of the temperature and the heat flux, respectively.

To analyze the transmission lines in the frequency domain it is considered the Fourier transform operator F such that $I(x,j\omega) = F\{i(x,t)\}$ and $V(x,i\omega) = F\{v(x,t)\}$ (with $j = (-1)^{1/2}$) and equations (8) are transformed to:

$$\frac{dV(x, j\omega)}{dx} = -Z(i\omega)I(x, j\omega) \tag{10a}$$

$$\frac{dI(x, j\omega)}{dx} = -Y(i\omega)V(x, j\omega) \tag{10b}$$

where $Z(j\omega) = R + j\omega L$ and $Y(j\omega) = G + j\omega C$. In the same line of thought equations (9) are transformed to:

$$\frac{d^2V(x, j\omega)}{dx^2} = -Z(i\omega)Y(i\omega)V(x, j\omega) \tag{11}$$

which as a solution in the frequency domain of the type:

$$V(x, j\omega) = A_1 e^{\gamma x} + A_2 e^{-\gamma x} \tag{12a}$$

$$I(x, j\omega) = Z_c^{-1}(A_2 e^{-\gamma x} - A_1 e^{\gamma x}) \tag{12b}$$

where $Z_c(j\omega) = [Z(j\omega)Y^{-1}(j\omega)]^{1/2} = [(R + j\omega L)/(G + j\omega C)]^{1/2}$ (characteristic impedance) and $\gamma(j\omega) = [Z(j\omega)Y(j\omega)]^{1/2} = \alpha(\omega) + j\beta(\omega)$.

These expressions have two terms corresponding to waves traveling in opposite directions: the term proportional to $e^{-\gamma x}$ is due to the signal applied at the line input while the term $e^{\gamma x}$ represents the reflected wave.

For a transmission line of length l it is usual to adopt as variable the distance up to the end given by:

$$y = l - x \tag{13}$$

If V_2 and I_2 represent the voltage and current at the end of the transmission line then the Fourier transforms of equations (8) at coordinate y are given by:

$$V(x, j\omega) = V_2 ch(\gamma y) + I_2 Z_c sh(\gamma y) \tag{14a}$$

$$I(x, j\omega) = V_2 Z_c^{-1} sh(\gamma y) + I_2 ch(\gamma y) \tag{14b}$$

Therefore, for a loading impedance $Z_2(j\omega)$ we have $V_2(j\omega) = Z_2(j\omega)I_2(j\omega)$ and the input impedance $Z_i(j\omega)$ of the transmission line results:

$$Z_i(j\omega) = [Z_2 ch(\gamma y) + Z_c sh(\gamma y)] \left[Z_2 Z_c^{-1} sh(\gamma y) + ch(\gamma y)\right]^{-1} \tag{15}$$

Typically are considered three cases at the end of the line, namely a short circuit, an open circuit and an adapted line, that simplify equation (15) yielding:

$$V_2 = 0, Z_2(j\omega) = 0, Z_i(j\omega) = Z_c(j\omega)th(\gamma l) \tag{16a}$$

$$I_2 = 0, Z_2(j\omega) = \infty, Z_i(j\omega) = Z_c(j\omega)cth(\gamma l) \tag{16b}$$

$$Z_2(j\omega) = Z_c(j\omega), Z_i(j\omega) = Z_c(j\omega) \tag{16c}$$

The classical perspective is to study lossless lines (*i.e.*, $R = 0$ and $G = 0$), reasonable in power systems, and approximations in the frequency domain leading to two-port networks with integer order elements. This is surprising because the transcendental equations (15) and (16) may lead both to integer and fractional-order expressions. For example, in the case of an adapted line (with R, C, L, $G \in \Re^+$), we can have from half-order fractional capacitances up to half-order fractional inductances, that is $-\pi/4 \le \arg\{Z_c(j\omega)\} \le \pi/4$, according with the expressions:

$$L = 0 \ and \ G = 0 \Rightarrow Z_c(j\omega) = \left[(j\omega)^{-1}RC^{-1}\right]^{1/2} \tag{17a}$$

$$R = kL \ and \ G = kC \ (k \in \Re^+) \Rightarrow Z_c(j\omega) = (RL^{-1})^{1/2} \tag{17b}$$

$$R = 0 \ and \ C = 0 \ \Rightarrow \ Z_c(j\omega) = (j\omega LG^{-1})^{1/2} \tag{17c}$$

These results, overlooked in the (integer-order point of view) classical textbooks, suggest possible strategies for implementing fractional-order impedances, somehow as standard microstrips and striplines work in microwave circuits. In fact, this hardware strategy of implementing fractional-order derivatives has been recently pointed out in order to avoid computational approximation schemes [9, 10]. Therefore, an alternative to exploring fractal geometries [11] and dielectric properties [12] to achieve fractional capacitors we can also turn our attention to the distributed characteristics of this type of system in order to design integrated circuits capable of implementing directly fractional derivatives.

3.2 Heat Diffusion

In many industrial applications it is important that the temperature distribution in the work pieces should be as uniform as possible. It is clearly difficult to determine the temperature distribution in the interior of the material or system, but the measurement of the surface temperature is routine. Therefore, we encounter the problem of the observability and control of the temperature distribution throughout the material from the available surface measurements.

The heat diffusion [13, 14] is represented by a linear partial differential equation (*PDE*):

$$\frac{\partial u}{\partial t} = k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \tag{18}$$

where k is the diffusivity, t is the time, u is the temperature and (x,y,z) are the space cartesian coordinates.

This system involves the integration of a *PDE* of parabolic type for which the standard theory of parabolic *PDE*s guarantees the existence of a unique solution.

For the case of a planar perfectly isolated surface we apply a constant temperature U_0 at $x = 0$ and we analyze the heat diffusion along horizontal coordinate x. The heat diffusion, under the previous conditions, is characterized by a model of non-integer order. In fact, the *PDE* solution in the s domain corresponds to the expression:

$$U(x, s) = \frac{U_0}{s} G(s), \quad G(s) = e^{-x\sqrt{\frac{s}{k}}} \tag{19}$$

where x is the space coordinate and U_0 is the boundary condition.

The corresponding solution in the time domain yields:

$$u(x, t) = U_0 erfc\left(\frac{x}{2\sqrt{kt}} \right) = U_0 \left(1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{kt}}} e^{-u^2} du \right) \tag{20}$$

In our study we adopt the Crank-Nicolson implicit numerical integration based on the discrete approximation to differentiation, yielding the equation:

$$- ru[j + 1, i + 1] + (2 + r) u[j + 1, i] - ru[j + 1, i - 1] =$$

$$ru[j, i + 1] + (2 - r) u[j, i] + ru[j, i - 1] \tag{21}$$

where $r = k\Delta t (\Delta x^2)^{-1}$, $\{\Delta x, \Delta t\}$ and $\{i, j\}$ are the increments and integration indices for space and time, respectively.

We verify that the results obtained through the numerical approach differ from the analytical results for low frequencies. This is illustrated in Figure 6, which depicts the polar diagram for $x = 3.0$ m and $k = 0.042$ m^{2}s^{-1}, both for the theoretical $G(j\omega)$ and the numerical $\widetilde{G}(j\omega)$ methods.
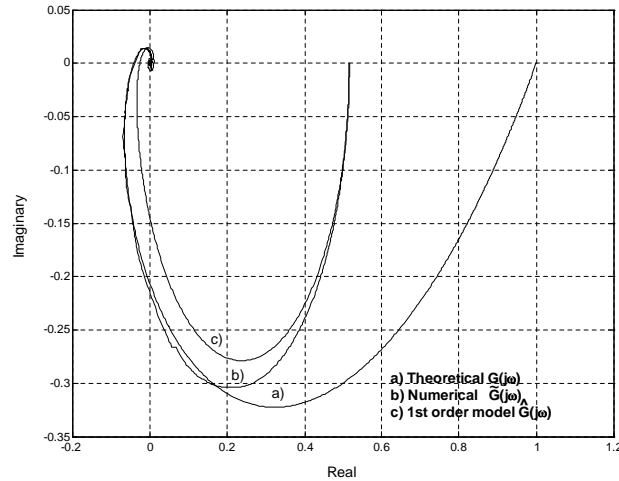
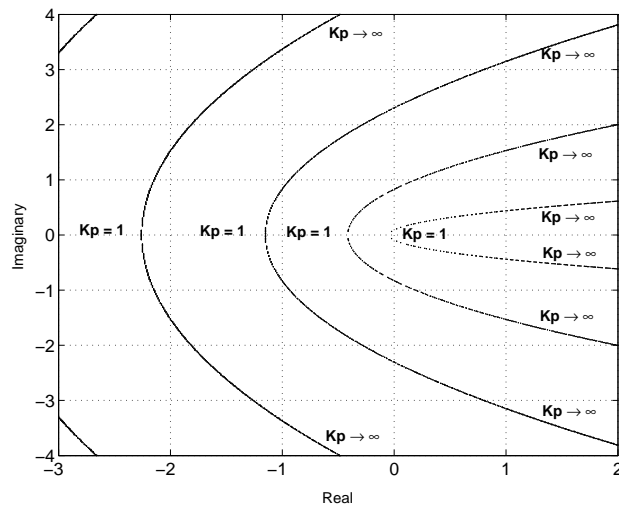Figure 6: Open loop polar diagrams of $G(j\omega)$, $\widetilde{G}(j\omega)$ and $\hat{G}(j\omega)$ for $x = 3.0$ m, $k = 0.042$ m^{2}s^{-1}

Figure 7: Root-locus of $G(s) = e^{-x\sqrt{\frac{s}{k}}}$ with $G_c(s) = k_p$, $x = 3.0$ m, $k = 0.042$ m^{2}s^{-1}

The open loop polar diagram has similarities to those of a low pass filter with time-delay. In this line of thought, we consider the control of the heat system with three algorithms. In a first phase (Figure 9a) we analyze a simple proportional controller ($G_c(s) = k_p$), with the root-locus of Figure 7, tuned through the Ziegler-Nichols open loop (*ZNOL*) method. The *ZNOL* heuristics leads to the approximate system model $\hat{G}(s) = k_B e^{-sT} / (s\tau + 1)$ with $\{k_B, T, \tau\} = \{0.52, 0.165, 1.235\}$ and the P controller parameter $k_p = 14.516$. Figure 8 depicts the corresponding time response for $R(s) = 1/s$ and $x = 3.0$ m.

It is clear that the P controller reveals a significant error in the steady state response.

In a second phase we adopt a *PID* controller ($G_c(s) = k_p [1+sT_d+(sT_i)^{-1}]$) also tuned through the *ZNOL* heuristics. Therefore, we have an identical approximate model $\hat{G}(s)$ leading to the *PID* parameters $\{k_p, T_d, T_i\} = \{0.3484, 0.0825, 0.33\}$. Figure 10 depicts the step response of the closed-loop system for $R(s) = 1/s$ and $x = 3.0$ m, that reveal a good steady-state response but a poor transient.
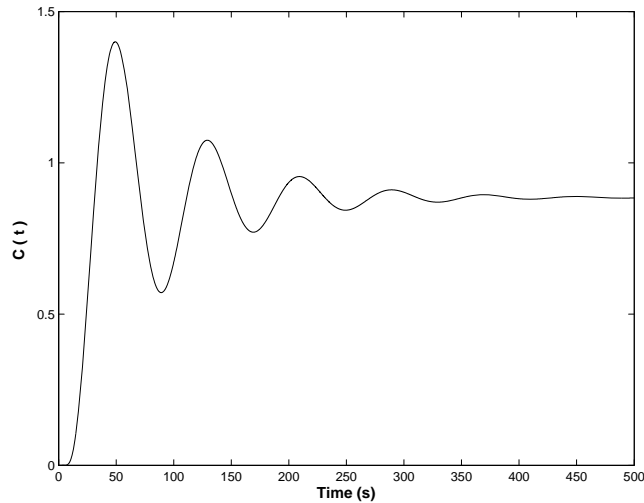
Figure 8: Time response for $R(s) = 1/s$ and $k_p = 14.516$

In a third phase we adopt a *PID* controller inserted in the Smith predictor structure (Figure 9b). This algorithm a well-known dead-time compensation technique that is very effective in improving the control of processes having time delays. Figure 10 shows the corresponding time response for $R(s) = 1/s$.

It is clear that the Smith predictor leads to the best performance, both in the transient and the steady-state periods, revealing that we can adopt with success classical control algorithms in fractional-order dynamical systems.

4   Conclusions

This paper presented the fundamental aspects of the theory of *FC*, the main approximation methods for the fractional-order derivatives calculation and the implication of the *FC* concepts on the extension of the classical systems theory. Bearing these ideas in mind, two distributed parameter linear systems were described and their dynamics was analyzed in the perspective of fractional calculus. It was shown that fractional-order models capture phenomena and properties that classical integer-order simply neglect. In this line of thought, this article is a step towards the development of systems modeling and control based on the theory of *FC*.

References

[1] K. B. Oldham and J. Spanier. *The Fractional Calculus*. Academic Press, 1974.

[2] K. S. Miller and B. Ross. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. John Wiley & Sons, 1993.
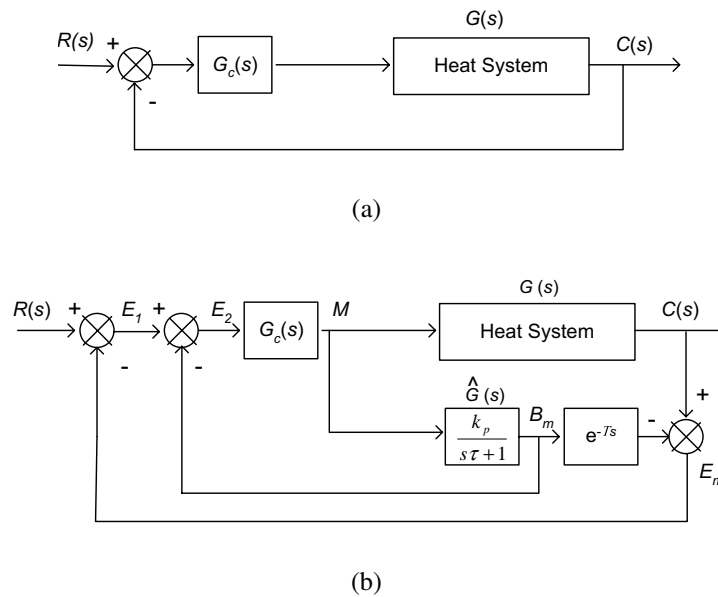
(a)

(b)

Figure 9: Block diagram of closed-loop system with a) P or PID b) Smith predictor with PID.

[3] I. Podlubny. *A Fractional Differential Equations*. Academic Press, 1999.

[4] A. Oustaloup. *La Commande CRONE: Commande Robuste d'Ordre Non Entier*. Hermes, 1991.

[5] J. A. Tenreiro Machado. The Analysis and Design of Fractional-Order Digital Control Systems. *SAMS Journal Systems Analysis, Modelling, Simulation*, pages 107–122, 1997.

[6] Olivier Heaviside. *Electromagnetic Theory*. 1893.

[7] Richard B. Adler, Lan Jen Chu, and Robert M. Fano. *Electromagnetic Energy Transmission and Radiation*. M.I.T. Press, 1969.

[8] Robert A. Chipman. *Transmission Lines*. McGraw-Hill, 1968.

[9] Alain Le Méhauté, Raoul R. Nigmatullin, and Laurent Nivanen. *Flèches du Temps et Géométrie Fractale*. Edition Hermes, 1998.

[10] Svante Westerlund. *Dead Matter has Memory!* Kalmar, 2002.

[11] Hirad Samavati, Ali Hajimiri, Arvin Shahani, Gitty N. Nasserbakht, and Thomas H. Lee. The Fractal Capacitors. *IEEE Journal of Solid-State Circuits*, pages 2035–2041, 1998.

[12] Gary W. Bohannan. Analog realization of a fractional control element - revisited. In *Wavelength Electronics, Inc*, pages 175–182, 2002.

[13] Jean-Luc Battaglia, Ludivic Le Ly, J. Batsale, A. Oustaloup, and O. Cois. The Utilisation de Modèles d'Identification non Entiers pour la Résolution de Problèmes Inverses en Conduction. *Int. Journal Therm. Science*, pages 374–389, 2000.

[14] J. Tenreiro Machado, Ramiro Barbosa, Manuel Silva, Lino Figueiredo, Isabel Jesus, and Alexandra Galhano. Application of fractional calculus in the system modelling and control. In *Proceedings of the ICKEDS'2004 – Int. Conf. on Knowledge Engineering and Decision Support*, pages 445–452, July 2004.
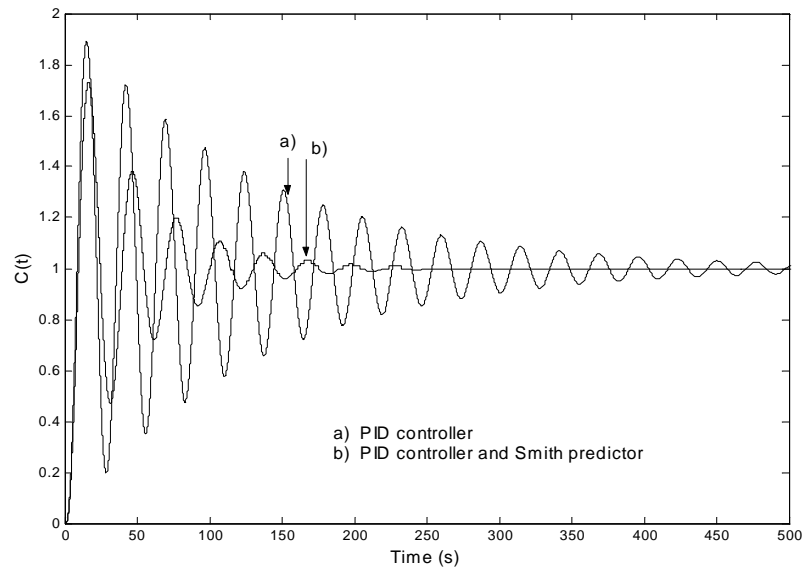
Figure 10: Step response of closed-loop system for $k_p = 0.3484$, $T_d = 0.0825$s and $T_i = 0.33$s

About the Authors

J. A. Tenreiro Machado was born in October 6, 1957. He graduated and received the Ph.D. degree in electrical and computer engineering from the Faculty of Engineering of the University of Porto, Portugal, in 1980 and 1989, respectively. Presently he is Coordinator Professor at the Institute of Engineering of the Polytechnic Institute of Porto, Department of Electrical Engineering. His main research interests are robotics, modelling, control, genetic algorithms, fractional-order systems and intelligent transportation systems.

Isabel S. Jesus was born in November 14, 1967. She graduated in Electrical Engineering - Industrial Control from Institute of Engineering of Polytechnic Institute of Porto, Portugal, in 1999 and received the Master's degree in Electrical and Computer Engineering from the Faculty of Engineering of the University of Porto, Portugal, in 2002. Presently she teaches at the Institute of Engineering of the Polytechnic Institute of Porto, Department of Electrical Engineering. Her research interests include modelling, control, fractional-order systems, and nonlinear systems.

J. Boaventura Cunha was born in November 21, 1961. He graduated in Electronics Engineering from the University of Aveiro in 1985 and received the Ph.D. degree in electrical and computer engineering from the University of Trás-os-Montes e Alto Douro, Portugal, in 2002. Presently he is an Assistant Professor at the Engineering Department of the University of Trás-os-Montes e Alto Douro and he is a researcher at the CETAV Institute. His main research interests are automation, modelling and control systems.

József K. Tar received his Master's degree in Physics in 1981 from the Loránd Eötvös University, and PhD degree in 1989 in Budapest, Hungary. His research focuses on the adaptive control of very inaccurately modelled, dynamically coupled, non-linear systems.